

The Renormalization Group According to Balaban

II. Large Fields

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Abstract

This is an expository account of Balaban's approach to the renormalization group. The method is illustrated with a treatment of the the ultraviolet problem for the scalar ϕ^4 model on toroidal lattice in dimension $d = 3$. In this second paper we control the large field contribution to the partition function

1 Introduction

This paper is an extension of part I [19]. We recall the general setup. We are studying the ϕ^4 field theory on a toroidal lattice of the form

$$\mathbb{T}_M^{-N} = (L^{-N}\mathbb{Z}/L^M\mathbb{Z})^3 \quad (1)$$

The theory is scaled up to the unit lattice \mathbb{T}_{M+N}^0 and there the partition function has the form

$$Z_{M,N} = \int \rho_0^N(\Phi) d\Phi \quad (2)$$

where for fields $\Phi : \mathbb{T}_{M+N}^0 \rightarrow \mathbb{R}$ we have the density

$$\rho_0^N(\Phi) = \exp(-S_0^N(\Phi) - V_0^N(\Phi)) \quad (3)$$

with

$$\begin{aligned} S_0^N(\Phi) &= \frac{1}{2} \|\partial\Phi\|^2 + \frac{1}{2} \bar{\mu}_0^N \|\Phi\|^2 \\ V_0^N(\Phi) &= \varepsilon_0^N \text{Vol}(\mathbb{T}_{M+N}) + \frac{1}{2} \mu_0^N \|\Phi\|^2 + \frac{1}{4} \lambda_0^N \sum_x \Phi^4(x) \end{aligned} \quad (4)$$

and very small coupling constants $\lambda_0^N = L^{-N}\lambda$, $\mu_0^N = L^{-2N}\mu$, etc. The superscript N is generally omitted so we have λ_0, μ_0 , etc..

Our goal is to show that with intelligent choices of the counter terms ε_0^N, μ_0^N the partition function $Z_{M,N}$ satisfies stability bounds which are uniform in the ultraviolet cutoff N and with bulk dependence on the volume parameter M . The method is the renormalization group method of Balaban ([1] - [14]).

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In fact our primary goal is not the stability bounds, which are interesting but not new, but rather the illustration of Balaban's method.

The first renormalization group operation is defined as follows. We create a new density defined for $\Phi_1 : \mathbb{T}_{M+N}^1 \rightarrow \mathbb{R}$ by

$$\tilde{\rho}_1(\Phi_1) = \text{const} \int \exp\left(-\frac{1}{2} \frac{a}{L^2} \|\Phi_1 - Q\Phi_0\|^2\right) \rho_0(\Phi_0) d\Phi_0 \quad (5)$$

Here Q averages over blocks of linear size L . Then one scales back to a unit lattice replacing Φ_1 by $\Phi_{1,L}$ where now $\Phi_1 : \mathbb{T}_{M+N-1}^0 \rightarrow \mathbb{R}$ and

$$\Phi_{1,L}(x) = L^{-1/2} \Phi(x/L) \quad (6)$$

Thus we define

$$\rho_1(\Phi_1) = \text{const} \tilde{\rho}_1(\Phi_{1,L}) \quad (7)$$

The constants are chosen to preserve the integral:

$$\int \rho_1(\Phi_1) d\Phi_1 = \int \rho_0(\Phi_0) d\Phi_0 \quad (8)$$

This operation is repeated. However to control the densities $\rho_0, \rho_1, \rho_2, \dots$ that are generated we need an extensive analysis at each stage. The key idea is to analyze large and small field regions separately. We give a first taste here in a somewhat simplified version.

In (5) insert under the integral sign

$$1 = \sum_{\Omega_1} \zeta(\Omega_1^c, \Phi_0) \chi(\Omega_1, \Phi_0) \quad (9)$$

Here we have partitioned the lattice into cubes \square of linear size $M = L^m$. We are summing over regions Ω_1 which are unions of such cubes. The function $\chi(\Omega_1, \Phi_0)$ is the characteristic function of the set of fields which satisfy some small field conditions in Ω_1 . The conditions are

$$|\Phi_1 - Q\Phi_0| \leq p_0 \quad |\partial\Phi_0| \leq p_0 \quad |\Phi_0| \leq \lambda_0^{-\frac{1}{4}} p_0 \quad (10)$$

where $p_0 = (-\log \lambda_0)^p$ is actually rather large. The function $\zeta(\Omega_1^c, \Phi_0, \Phi_1)$ is the characteristic function of fields which violate at least one of these inequalities at some point in each cube in Ω_1^c .

The resulting integral can now be written (splitting the bonds across $\partial\Omega_1$)

$$\begin{aligned} \tilde{\rho}_1(\Phi_1) = & \text{const} \sum_{\Omega_1} \int d\Phi_{0,\Omega_1^c} \zeta(\Omega_1^c, \Phi_0) \exp\left(-\frac{1}{2} \frac{a}{L^2} \|\Phi_1 - Q\Phi_0\|_{\Omega_1^c}^2 - S_0(\Omega_1^c, \Phi_0) - V_0(\Omega_1^c, \Phi_0)\right) \\ & \left[\int d\Phi_{0,\Omega_1} \chi(\Omega_1, \Phi_0) \exp\left(-\frac{1}{2} \frac{a}{L^2} \|\Phi_1 - Q\Phi_0\|_{\Omega_1}^2 - S_0(\Omega_1, \Phi_0) - V_0(\Omega_1, \Phi_0)\right) \right] \end{aligned} \quad (11)$$

The idea is now to carry out a detailed analysis of the small field integral over $[\dots]$ This involves expanding around the field Φ_0 which minimizes the first two terms in the exponent. This generates a new action $S_1(\Omega_1, \Phi_1)$ and a fluctuation integral. Then one writes the fluctuation integral in a local form, which means doing a cluster expansion. After scaling the fields and the region Ω_1 we have a new contribution to the density of the form

$$[\dots] = \exp\left(-S_1(\Omega_1, \Phi_1) - V_1(\Omega_1, \Phi_1) + \sum_{X \subset \Omega_1} E_1(X, \Phi_1)\right) \quad (12)$$

The first two terms are similar to what we started with but now with new coupling constants $\lambda_1 = L\lambda_0$, $\mu_1 = L^2\mu_0 + \dots$, etc. The localized functions $E(X, \Phi_1)$ are defined for polymers X (connected unions

of M cubes), depend only on Φ_1 restricted to X , and are exponentially decaying in $|X|_M$ (the number of M cubes in X).

Now consider the large field region Ω_1^c . In a cube $\square \subset \Omega_1^c$ at least one of $\exp(-a/2L^2\|\Phi_1 - Q\Phi_0\|_{\square}^2)$ or $\exp(-S_0(\square, \Phi_0))$ or $\exp(-V_0(\square, \Phi_0))$ is bounded by $e^{-\mathcal{O}(1)p_0^2}$. Thus the first exponential in (11) is bounded by $e^{-\mathcal{O}(1)p_0^2|\Omega_1^c|_M}$. This tiny factor is sufficient to strongly suppress the contribution of any large field region and control the sum over Ω_1 .

Now one repeats this operation. At the k^{th} step we have a tighter definition of small fields based on the new larger coupling constants $\lambda_k = L^k \lambda_0$ and smaller parameters $p_k = (-\log \lambda_k)^p$. Correspondingly one makes a new large/small split Ω_{k+1} in the current small field region Ω_k . The overall result is a sum over decreasing regions $\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_k$ with an explicit leading action in the current small field region Ω_k .

The main issues are the detailed analysis of (1.) the small field region, (2.) the large field regions, (3.) the coupling between them, and (4.) the convergence of all the sums. Point (1.) was considered in detail in the first paper. Here we are concerned with points (2.) and (3.). A third and final paper establishes point (4.) and completes the proof of the stability bound.

Although the broad outlines of the procedure are due to Balaban, especially in his treatment of the linear sigma model [8] - [14], we deviate in many of the particulars.

Before plunging into the details of the general problem we open with an analysis of the free effective actions which are generated by this process

convention: Throughout the text $\mathcal{O}(1)$ stands for a constant independent of all other parameters. Also C stands for a constant depending on L , but on no other parameters. Both $\mathcal{O}(1)$ and C can change from line to line.

2 Localized block averaging

2.1 block averaging

First some definitions. In any of our lattices \mathbb{T}_{M+N-k}^{-k} the centers of L^n -cubes are the points in the lattice $\mathbb{T}_{M+N-k}^{-(k-n)}$. For a region $\Omega \subset \mathbb{T}_{M+N-k}^{-k}$, let $\Omega^{(n)}$ be the centers of L^n -cubes in Ω , thus we have $\Omega^{(n)} = \Omega \cap \mathbb{T}_{M+N-k}^{-(k-n)}$.

Let Φ_0 be a function on the initial torus \mathbb{T}_{M+N}^0 and $\rho_0(\Phi_0)$ an initial density. Let Ω_1 be a union of LM blocks in \mathbb{T}_{M+N}^0 for some large $M = L^m$. If Φ_{0,Ω_1} is the restriction of Φ_0 to Ω_1 and Q is averaging over L -cubes, then $Q\Phi_{0,\Omega_1}$ is a function on $\Omega_1^{(1)}$ ($= \Omega_1 \cap \mathbb{T}_{M+N}^1$). If Φ_{1,Ω_1} is any other function on $\Omega_1^{(1)}$ we can form ¹

$$\|\Phi_1 - Q\Phi_0\|_{\Omega_1^{(1)}}^2 \equiv L^3 \sum_{x \in \Omega_1^{(1)}} |\Phi_1(x) - Q\Phi_0(x)|^2 \equiv L^3 |\Phi_1 - Q\Phi_0|_{\Omega_1^{(1)}}^2 \quad (13)$$

In the following we generally write an expression like $\|\Phi_1 - Q\Phi_0\|_{\Omega_1^{(1)}}^2$ as just $\|\Phi_1 - Q\Phi_0\|_{\Omega_1}^2$ and the expression $|\Phi_1 - Q\Phi_0|_{\Omega_1^{(1)}}^2$ as just $|\Phi_1 - Q\Phi_0|_{\Omega_1}^2$. It is understood that the norm is to be evaluated on the intersection of the domain of the fields with Ω_1 , with the appropriate weighting.

¹In general if f is defined on $\Omega \subset \mathbb{T}^{-k}$ then

$$\|f\|_{\Omega}^2 \equiv L^{-3k} \sum_{x \in \Omega} |f(x)|^2 \equiv L^{-3k} |f|_{\Omega}^2$$

We define block averaging in Ω_1 by

$$\begin{aligned}\tilde{\rho}_{1,\Omega_1}(\Phi_{0,\Omega_1^c}, \Phi_{1,\Omega_1}) &= \mathcal{N}_{aL,\Omega_1^{(1)}}^{-1} \int \exp\left(-\frac{1}{2} \frac{a}{L^2} \|\Phi_1 - Q\Phi_0\|_{\Omega_1}^2\right) \rho_0(\Phi_0) d\Phi_{0,\Omega_1} \\ &= \mathcal{N}_{aL,\Omega_1^{(1)}}^{-1} \int \exp\left(-\frac{1}{2} aL |\Phi_1 - Q\Phi_0|_{\Omega_1}^2\right) \rho_0(\Phi_0) d\Phi_{0,\Omega_1}\end{aligned}\quad (14)$$

where

$$\mathcal{N}_{a,\Omega} = (2\pi/a)^{|\Omega|/2} \quad |\Omega| = \text{number of elements in } \Omega \quad (15)$$

The preserves the integral:

$$\int \tilde{\rho}_{1,\Omega_1}(\Phi_{0,\Omega_1^c}, \Phi_{1,\Omega_1}) d\Phi_{0,\Omega_1^c} d\Phi_{1,\Omega_1} = \int \rho_0(\Phi_0) d\Phi_0 \quad (16)$$

Next we scale back down to a unit lattice. Replace Ω_1 by $L\Omega_1$ where now Ω_1 is a union of M -blocks in \mathbb{T}_{M+N-1}^{-1} (i.e. length M , not M sites), so that $L\Omega_1$ is a union of LM blocks in \mathbb{T}_{M+N}^0 . Then replace $\Phi_{1,L\Omega_1}$, a function on $(L\Omega_1)^{(1)} = L\Omega_1 \cap \mathbb{T}_{M+N}^1$, by $[\Phi_{1,L}]_{L\Omega_1} = [\Phi_{1,\Omega_1}]_L$ where now Φ_{1,Ω_1} is a function on $\Omega_1^{(1)} = \Omega_1 \cap \mathbb{T}_{M+N-1}^0$. Also replace $[\Phi_0]_{L\Omega_1^c}$, a function on $L\Omega_1^c \subset \mathbb{T}_{M+N}^0$ by $[\phi_L]_{L\Omega_1^c} = [\phi_{\Omega_1^c}]_L$ where now $\phi_{\Omega_1^c}$ is a function on $\Omega_1^c \subset \mathbb{T}_{M+N-1}^{-1}$. Thus the definition is

$$\rho_{1,\Omega_1}(\phi_{\Omega_1^c}, \Phi_{1,\Omega_1}) = \text{const } \tilde{\rho}_{1,L\Omega_1}\left([\phi_{\Omega_1^c}]_L, [\Phi_{1,\Omega_1}]_L\right) \quad (17)$$

with the constant chosen to preserve the integral.

We iterate this procedure always shrinking the region in which we are averaging. (Later we impose some conditions on this shrinking). After k steps we will have a sequence of regions

$$\Omega = (\Omega_1, \Omega_2, \dots, \Omega_k) \quad (18)$$

in \mathbb{T}_{M+N-k}^{-k} which satisfy

$$\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_k \quad (19)$$

The region Ω_j is a union of $L^{-(k-j)}M$ blocks. We also define

$$\delta\Omega_j = \Omega_j - \Omega_{j+1} \quad j = 1, 2, \dots, k-1 \quad (20)$$

See figure 1 for an indication of how a piece of this might look in the case where the complements Ω_j^c are small (the more likely case).

After k steps we will have a density $\rho_{k,\Omega}(\phi_{\Omega_1^c}, \Phi_{k,\Omega})$ with the same integral. Here $\phi_{\Omega_1^c} : \Omega_1^c \rightarrow \mathbb{R}$ and

$$\Phi_{k,\Omega} = (\Phi_{1,\delta\Omega_1}, \Phi_{2,\delta\Omega_2}, \dots, \Phi_{k-1,\delta\Omega_{k-1}}, \Phi_{k,\Omega_k}) \quad (21)$$

where

$$\begin{aligned}\Phi_{j,\delta\Omega_j} &: \delta\Omega_j^{(j)} \rightarrow \mathbb{R} \quad j = 1, \dots, k-1 \\ \Phi_{k,\Omega_k} &: \Omega_k^{(k)} \rightarrow \mathbb{R}\end{aligned}\quad (22)$$

Note that $\delta\Omega_j^{(j)} \subset \mathbb{T}_{M+N-k}^{-(k-j)}$ and $\Omega_k^{(k)} \subset \mathbb{T}_{M+N-k}^0$. The fields $\Phi_{k,\Omega}$ can also be regarded as a single function on

$$\delta\Omega_1^{(1)} \cup \delta\Omega_2^{(2)} \cup \dots \cup \delta\Omega_{k-1}^{(k-1)} \cup \Omega_k^{(k)} \subset \mathbb{T}_{M+N-k}^{-k} \quad (23)$$

The next step is taken by introducing by choosing $\Omega_{k+1} \subset \Omega_k$ which is a union of LM blocks in \mathbb{T}_{M+N-k}^{-k} . There is a new field $\Phi_{k+1} : \Omega_{k+1}^{(k+1)} \rightarrow \mathbb{R}$ ($\Omega_{k+1}^{(k+1)} = \Omega_{k+1} \cap \mathbb{T}_{M+N-k}^1$) Now define

$$\begin{aligned}\Omega^+ &= (\Omega, \Omega_{k+1}) = (\Omega_1, \Omega_2, \dots, \Omega_k, \Omega_{k+1}) \\ \Phi_{k+1,\Omega^+} &= (\Phi_{1,\delta\Omega_1}, \dots, \Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}})\end{aligned}\quad (24)$$

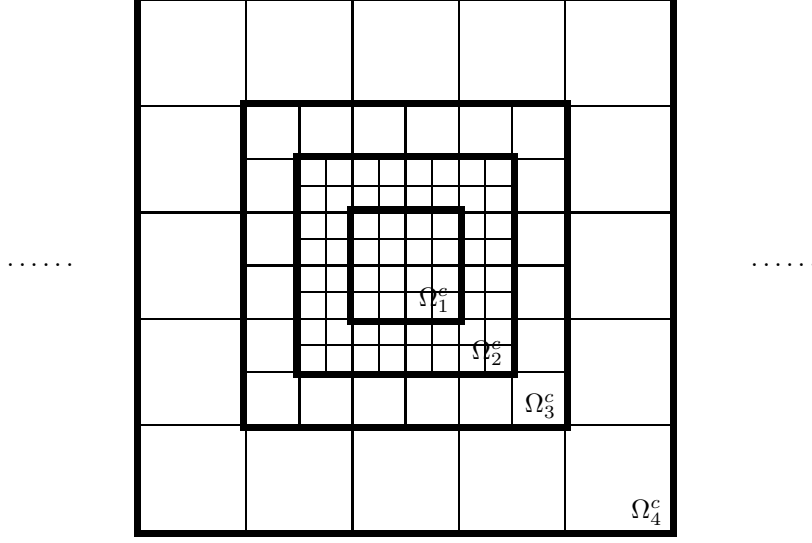


Figure 1: Nested regions $\Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \Omega_4$

and

$$\begin{aligned}
& \tilde{\rho}_{k+1, \mathbf{\Omega}^+}(\phi_{\Omega_1^c}, \Phi_{k+1, \mathbf{\Omega}^+}) \\
&= \mathcal{N}_{aL, \Omega_{k+1}^{(k+1)}}^{-1} \int \exp\left(-\frac{1}{2} \frac{a}{L^2} \|\Phi_{k+1} - Q\Phi_k\|_{\Omega_{k+1}}^2\right) \rho_{k, \mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_k, \mathbf{\Omega}) d\Phi_{k, \Omega_{k+1}} \\
&= \mathcal{N}_{aL, \Omega_{k+1}^{(k+1)}}^{-1} \int \exp\left(-\frac{1}{2} aL |\Phi_{k+1} - Q\Phi_k|_{\Omega_{k+1}}^2\right) \rho_{k, \mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_k, \mathbf{\Omega}) d\Phi_{k, \Omega_{k+1}}
\end{aligned} \tag{25}$$

Next we scale. Replace $\mathbf{\Omega}^+$ by $L\mathbf{\Omega}^+$ where still $\mathbf{\Omega}^+ = (\Omega_1, \Omega_2, \dots, \Omega_k, \Omega_{k+1})$ but now Ω_j is a union of $L^{-(k+1-j)}M$ blocks in $\mathbb{T}_{M+N-k-1}^{-k-1}$. Replace $\phi_{L\Omega_1^c}$ by $[\phi_L]_{L\Omega_1^c} = [\phi_{\Omega_1^c}]_L$ where ϕ is defined on the new Ω_1^c . For $1 \leq j \leq k$ replace $\Phi_{j, L\delta\Omega_j}$ by $[\Phi_{j, L}]_{L\delta\Omega_j} = [\Phi_{j, \delta\Omega_j}]_L$ where now $\Phi_{j, \delta\Omega_j}$ is defined on $\delta\Omega_j^{(j)} = \delta\Omega_j \cap \mathbb{T}_{M+N-k-1}^{-k-j-1}$, and similarly replace $\Phi_{k+1, L\Omega_{k+1}}$ by $[\Phi_{k+1, L}]_{L\Omega_{k+1}} = [\Phi_{k+1, \Omega_{k+1}}]_L$ where now Φ_{k+1} is defined on $\Omega_{k+1}^{(k+1)} = \Omega_{k+1} \cap \mathbb{T}_{M+N-k-1}^0$. With these changes we get the a function of the new $\Phi_{k+1, \mathbf{\Omega}^+} = (\Phi_{1, \delta\Omega_1}, \dots, \Phi_{k, \delta\Omega_k}, \Phi_{k+1, \Omega_{k+1}})$ defined by

$$\rho_{k+1, \mathbf{\Omega}^+}(\phi_{\Omega_1^c}, \Phi_{k+1, \mathbf{\Omega}^+}) = \text{const } \tilde{\rho}_{k+1, L\mathbf{\Omega}^+}([\phi_{\Omega_1^c}]_L, [\Phi_{k+1, \mathbf{\Omega}^+}]_L) \tag{26}$$

We can also compose the various averaging operators. For any field on any lattice define $Q_j\Phi = Q^j\Phi$. This is averaging over cubes with L^j sites on a side. Then define for ϕ on \mathbb{T}_{M+N-k}^{-k}

$$Q_{k, \mathbf{\Omega}}\phi = ([Q_1\phi]_{\delta\Omega_1}, \dots, [Q_{k-1}\phi]_{\delta\Omega_{k-1}}, [Q_k\phi]_{\Omega_k}) \tag{27}$$

where $[Q_j\phi]_{\delta\Omega_j} : \delta\Omega_j^{(j)} \rightarrow \mathbb{R}$. Let $\Phi_{k, \mathbf{\Omega}}$ be any other field as in (21) and define

$$\mathbf{a} = \mathbf{a}^{(k)} = (a_1^{(k)}, \dots, a_k^{(k)}) \quad a_j^{(k)} = a_j L^{2(k-j)} \tag{28}$$

Then we can form

$$\|\mathbf{a}^{\frac{1}{2}}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi)\|^2 \equiv \sum_{j=1}^{k-1} a_j^{(k)} \|\Phi_j - Q_j\phi\|_{\delta\Omega_j}^2 + a_k \|\Phi_k - Q_k\phi\|_{\Omega_k}^2 \quad (29)$$

Lemma 2.1. *For some constant $\mathcal{N}_{k,\mathbf{\Omega}}$*

$$\rho_{k,\mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_{k,\mathbf{\Omega}}) = \mathcal{N}_{k,\mathbf{\Omega}}^{-1} \int \exp\left(-\frac{1}{2}\|\mathbf{a}^{\frac{1}{2}}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi)\|^2\right) \rho_0(\phi_{L^k}) d\phi_{\Omega_1} \quad (30)$$

Proof. The statement for $k = 1$ follows from (14), (17). Suppose it is true for k . Then

$$\begin{aligned} & \tilde{\rho}_{k+1,\mathbf{\Omega}}^+(\phi_{\Omega_1^c}, \Phi_{k+1,\mathbf{\Omega}}^+) \\ &= \text{const} \int \exp\left(-\frac{1}{2}\frac{a}{L^2}\|\Phi_{k+1} - Q\Phi_k\|_{\Omega_{k+1}}^2 - \frac{1}{2}\|(\mathbf{a}^{(k)})^{\frac{1}{2}}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi)\|^2\right) \rho_0(\phi_{L^k}) d\Phi_{k,\Omega_{k+1}} d\phi_{\Omega_1} \end{aligned} \quad (31)$$

We make the split $\|\Phi_k - Q_k\phi\|_{\Omega_k}^2 = \|\Phi_k - Q_k\phi\|_{\Omega_{k+1}}^2 + \|\Phi_k - Q_k\phi\|_{\delta\Omega_k}^2$. To evaluate the integral over $\Phi_{k,\Omega_{k+1}}$ we expand around the minimizer in $\Phi_{k,\Omega_{k+1}}$ of

$$\frac{a}{2L^2}\|\Phi_{k+1} - Q\Phi_k\|_{\Omega_{k+1}}^2 + \frac{a_k}{2}\|\Phi_k - Q_k\phi\|_{\Omega_{k+1}}^2 \quad (32)$$

This is a problem already discussed in part I on the whole torus. The solution is the same here. The variational equation for Φ_k is

$$\left(a_k + \frac{a}{L^2}Q^T Q\right)\Phi_k = a_k Q_k\phi + \frac{a}{L^2}Q^T \Phi_{k+1} \quad (33)$$

The solution is $\Psi_k = \Psi_{k,\Omega_{k+1}} = \Psi_{k,\Omega_{k+1}}(\Phi_{k+1}, \phi)$ given by

$$\Psi_k = Q_k\phi - \frac{aL^{-2}}{a_k + aL^{-2}}Q^T Q_{k+1}\phi + \frac{aL^{-2}}{a_k + aL^{-2}}Q^T \Phi_{k+1} \quad (34)$$

A short calculation shows that the value of (32) at the minimum is

$$\frac{a}{2L^2}\|\Phi_{k+1} - Q\Psi_k\|_{\Omega_{k+1}}^2 + \frac{a_k}{2}\|\Psi_k - Q_k\phi\|_{\Omega_{k+1}}^2 = \frac{a_{k+1}}{2L^2}\|\Phi_{k+1} - Q_{k+1}\phi\|_{\Omega_{k+1}}^2 \quad (35)$$

Now in (31) write $\Phi_{k,\Omega_{k+1}} = \Psi_{k,\Omega_{k+1}} + Z$ (all functions on the unit lattice $\Omega_{k+1}^{(k)}$) and integrate over Z instead of $\Phi_{k,\Omega_{k+1}}$. The terms with no Z 's are (35), the terms linear in Z vanish, and the terms quadratic in Z when integrated over Z yield a constant. Thus we have

$$\begin{aligned} & \tilde{\rho}_{k+1,\mathbf{\Omega}}^+(\phi_{\Omega_1^c}, \Phi_{k+1,\mathbf{\Omega}}^+) \\ &= \text{const} \int \exp\left(-\frac{a_{k+1}}{2L^2}\|\Phi_{k+1} - Q_{k+1}\phi\|_{\Omega_{k+1}}^2 - \frac{1}{2}\sum_{j=1}^k a_j^{(k)} \|\Phi_j - Q_j\phi\|_{\delta\Omega_j}^2\right) \rho_0(\phi_{L^k}) d\phi_{\Omega_1} \end{aligned} \quad (36)$$

Scaling as in (26) yields

$$\rho_{k+1,\mathbf{\Omega}}^+(\phi_{\Omega_1^c}, \Phi_{k+1,\mathbf{\Omega}}^+) = \text{const} \int \exp\left(-\frac{1}{2}\|(\mathbf{a}^{(k+1)})^{\frac{1}{2}}(\Phi_{k+1,\mathbf{\Omega}}^+ - Q_{k+1,\mathbf{\Omega}}^+\phi)\|^2\right) \rho_0(\phi_{L^{k+1}}) d\phi_{\Omega_1} \quad (37)$$

which is the result we want. The constant $\mathcal{N}_{k,\mathbf{\Omega}}^{-1}$ can be evaluated by integrating over all fields.

2.2 free flow

Now suppose we start with the free density given by (3) with $V_0 = 0$. Written in scaled form for $\phi : \mathbb{T}_{M+N-k}^{-k} \rightarrow \mathbb{R}$ it is

$$\rho_0(\phi_{L^k}) = \exp \left(-\frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k) \phi \rangle \right) \quad (38)$$

Thus we wish to evaluate

$$\rho_{k,\mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_{k,\mathbf{\Omega}}) = \mathcal{N}_{k,\mathbf{\Omega}}^{-1} \int \exp \left(-\frac{1}{2} \|\mathbf{a}^{\frac{1}{2}}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi)\|^2 - \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k) \phi \rangle \right) \quad (39)$$

The analysis depends on the decomposition

$$\begin{aligned} & \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k) \phi \rangle \\ &= \frac{1}{2} \langle \phi_{\Omega_1^c}, [-\Delta + \bar{\mu}_k]_{\Omega_1^c} \phi_{\Omega_1^c} \rangle + \langle \phi_{\Omega_1}, [-\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \rangle + \frac{1}{2} \langle \phi_{\Omega_1}, [-\Delta + \bar{\mu}_k]_{\Omega_1} \phi_{\Omega_1} \rangle \end{aligned} \quad (40)$$

Here ϕ_{Ω} is the restriction to Ω , $[-\Delta]_{\Omega} \equiv 1_{\Omega}[-\Delta]1_{\Omega}$ is the Laplacian with Dirichlet boundary conditions, and $[-\Delta]_{\Omega, \Omega^c} \equiv 1_{\Omega}[-\Delta]1_{\Omega^c}$.

Theorem 2.1. *Starting with the free density after k steps the density has the form*

$$\begin{aligned} & \rho_{k,\mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_{k,\mathbf{\Omega}}) \\ &= Z_{k,\mathbf{\Omega}} \exp \left(-\frac{1}{2} \|\mathbf{a}^{1/2}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi)\|_{\Omega_1}^2 - \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k) \phi \rangle \right) \quad \text{at} \quad \phi_{\Omega_1} = \phi_{k,\mathbf{\Omega}} \end{aligned} \quad (41)$$

Here $Z_{k,\mathbf{\Omega}}$ is a constant and $\phi_{k,\mathbf{\Omega}} : \mathbb{T}_{M+N-k}^{-k} \cap \Omega_1 \rightarrow \mathbb{R}$ is defined by

$$\phi_{k,\mathbf{\Omega}} = \phi_{k,\mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_{k,\mathbf{\Omega}}) = G_{k,\mathbf{\Omega}} \left(Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \right) \quad (42)$$

where

$$G_{k,\mathbf{\Omega}} = \left[-\Delta + \bar{\mu}_k + Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}} \right]_{\Omega_1}^{-1} \quad (43)$$

Proof. Inserting (40) into (39) we have

$$\begin{aligned} & \rho_{k,\mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_{k,\mathbf{\Omega}}) = \exp \left(-\frac{1}{2} \langle \phi_{\Omega_1^c}, [-\Delta + \bar{\mu}_k]_{\Omega_1^c} \phi_{\Omega_1^c} \rangle \right) \mathcal{N}_{k,\mathbf{\Omega}}^{-1} \\ & \int \exp \left(-\frac{1}{2} \|\mathbf{a}^{\frac{1}{2}}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi_{\Omega_1})\|^2 - \langle \phi_{\Omega_1}, [-\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \rangle - \frac{1}{2} \langle \phi_{\Omega_1}, [-\Delta + \bar{\mu}_0]_{\Omega_1} \phi_{\Omega_1} \rangle \right) d\phi_{\Omega_1} \end{aligned} \quad (44)$$

We do the integral by minimizing the exponent in ϕ_{Ω_1} . Taking the derivative in this variable and setting it equal to zero gives the variational equation

$$[-\Delta + \bar{\mu}_k + Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}}]_{\Omega_1} \phi_{\Omega_1} = Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \quad (45)$$

and the solution is $\phi_{\Omega_1} = \phi_{k,\mathbf{\Omega}}$ as given by (42).

Now in the exponent in (44) write $\phi_{\Omega_1} = \phi_{k,\mathbf{\Omega}} + \mathcal{Z}$ and integrate over \mathcal{Z} instead of ϕ_{Ω_1} . The term with no \mathcal{Z} 's comes outside the integral and gives the exponential in (41). The term linear in \mathcal{Z} vanishes. The term quadratic in \mathcal{Z} is $-\frac{1}{2} \langle \mathcal{Z}, [-\Delta + \bar{\mu}_k + Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}}]_{\Omega_1} \mathcal{Z} \rangle$. Thus we have the result with

$$Z_{k,\mathbf{\Omega}} = \mathcal{N}_{k,\mathbf{\Omega}}^{-1} \int \exp \left(-\frac{1}{2} \langle \mathcal{Z}, [-\Delta + \bar{\mu}_k + Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}}]_{\Omega_1} \mathcal{Z} \rangle \right) d\mathcal{Z} \quad (46)$$

Remark. The result can also be written in the form

$$\begin{aligned} & \rho_{k,\mathbf{\Omega}}(\phi_{\Omega_1^c}, \Phi_{k,\mathbf{\Omega}}) \\ &= Z_{k,\mathbf{\Omega}} \exp \left(-\frac{1}{2} \langle \phi_{\Omega_1^c}, [-\Delta + \bar{\mu}_k]_{\Omega_1^c} \phi_{\Omega_1^c} \rangle - \langle \phi_{k,\mathbf{\Omega}}, [-\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \rangle - S_k(\Omega_1, \Phi_{k,\mathbf{\Omega}}, \phi_{k,\mathbf{\Omega}}) \right) \end{aligned} \quad (47)$$

where

$$S_k(\Omega_1, \Phi_{k,\mathbf{\Omega}}, \phi) = \frac{1}{2} \|\mathbf{a}^{1/2}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi)\|_{\Omega_1}^2 + \frac{1}{2} \langle \phi, [-\Delta + \bar{\mu}_k]_{\Omega_1} \phi \rangle \quad (48)$$

This action can also be usefully written in terms of the fundamental variables:

Lemma 2.2.

$$S_k(\Omega_1, \Phi_{k,\mathbf{\Omega}}, \phi_{k,\mathbf{\Omega}}) = \frac{1}{2} \langle \Phi_{k,\mathbf{\Omega}}, \Delta_{k,\mathbf{\Omega}} \Phi_{k,\mathbf{\Omega}} \rangle + \frac{1}{2} \left\langle \phi_{\Omega_1^c}, [\Delta]_{\Omega_1^c, \Omega_1} G_{k,\mathbf{\Omega}} [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \right\rangle \quad (49)$$

where

$$\Delta_{k,\mathbf{\Omega}} = \mathbf{a} - \mathbf{a} Q_{k,\mathbf{\Omega}} G_{k,\mathbf{\Omega}} Q_{k,\mathbf{\Omega}}^T \mathbf{a} \quad (50)$$

Proof. From (48) we have

$$\begin{aligned} S_k(\Omega_1, \Phi_{k,\mathbf{\Omega}}, \phi_{k,\mathbf{\Omega}}) &= \frac{1}{2} \|\mathbf{a}^{1/2} \Phi_{k,\mathbf{\Omega}}\|^2 - \langle Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}}, \phi_{k,\mathbf{\Omega}} \rangle \\ &\quad - \frac{1}{2} \left\langle \phi_{k,\mathbf{\Omega}}, \left[-\Delta + \bar{\mu}_k + Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}} \right]_{\Omega_1} \phi_{k,\mathbf{\Omega}} \right\rangle \end{aligned} \quad (51)$$

Insert the expression for $\phi_{k,\mathbf{\Omega}}$ and obtain

$$\begin{aligned} S_k(\Omega_1, \Phi_{k,\mathbf{\Omega}}, \phi_{k,\mathbf{\Omega}}) &= \frac{1}{2} \|\mathbf{a}^{1/2} \Phi_{k,\mathbf{\Omega}}\|^2 - \left\langle Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}}, G_{k,\mathbf{\Omega}} \left(Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \right) \right\rangle \\ &\quad + \frac{1}{2} \left\langle \left(Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \right), G_{k,\mathbf{\Omega}} \left(Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \right) \right\rangle \\ &= \frac{1}{2} \|\mathbf{a}^{1/2} \Phi_{k,\mathbf{\Omega}}\|^2 - \frac{1}{2} \left\langle Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}}, G_{k,\mathbf{\Omega}} Q_{k,\mathbf{\Omega}}^T \mathbf{a} \Phi_{k,\mathbf{\Omega}} \right\rangle \\ &\quad + \frac{1}{2} \left\langle \phi_{\Omega_1^c}, [\Delta]_{\Omega_1^c, \Omega_1} G_{k,\mathbf{\Omega}} [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \right\rangle \end{aligned} \quad (52)$$

which is (49).

2.3 free flow - single step

Now we investigate how to follow the free flow a step at a time. This is in preparation for a similar step for the full model. Assuming the representation (41) in the next step we would want to compute (ignoring constants)

$$\int \exp \left(-J_{\mathbf{\Omega}^+}(\Phi_{k+1}, \Phi_{k,\mathbf{\Omega}}, (\phi_{\Omega_1^c}, \phi_{k,\mathbf{\Omega}})) \right) d\Phi_{k,\Omega_{k+1}} \quad (53)$$

where for $\phi : \mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k} \rightarrow \mathbb{R}$ and $\Phi_{k+1} : \Omega_{k+1}^{(k+1)} \rightarrow \mathbb{R}$

$$\begin{aligned} & J_{\mathbf{\Omega}^+}(\Phi_{k+1}, \Phi_{k,\mathbf{\Omega}}, \phi) \\ &= \frac{1}{2} \frac{a}{L^2} \|\Phi_{k+1} - Q_{k,\mathbf{\Omega}} \phi\|_{\Omega_{k+1}}^2 + \frac{1}{2} \|\mathbf{a}^{1/2}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}} \phi)\|_{\Omega_1}^2 + \frac{1}{2} \left\langle \phi, (-\Delta + \bar{\mu}_k) \phi \right\rangle \end{aligned} \quad (54)$$

To evaluate this integral we need to find the minimizer of $J_{\mathbf{\Omega}^+}(\Phi_{k+1}, \Phi_{k,\mathbf{\Omega}}, (\phi_{\Omega_1^c}, \phi_{k,\mathbf{\Omega}}))$ in $\Phi_{k,\Omega_{k+1}}$. Since this function is the minimum of $J_{\mathbf{\Omega}^+}(\Phi_{k+1}, \Phi_{k,\mathbf{\Omega}}, \phi)$ in ϕ_{Ω_1} , we can proceed by finding the minimum of $J_{\mathbf{\Omega}^+}(\Phi_{k+1}, \Phi_{k,\mathbf{\Omega}}, \phi)$ simultaneously in $\phi_{\Omega_1}, \Phi_{k,\Omega_{k+1}}$.

Lemma 2.3.

1. The unique minimum of $J_{\Omega^+}(\Phi_{k+1}, \Phi_{k,\Omega}, \phi)$ in $\phi_{\Omega_1}, \Phi_{k,\Omega_{k+1}}$ comes at $\phi_{\Omega_1} = \phi_{k+1,\Omega^+}^0$ where ²

$$\phi_{k+1,\Omega^+}^0(\phi_{\Omega_1^c}, \Phi_{k+1,\Omega^+}) = G_{k+1,\Omega^+}^0 \left(L^{-2} Q_{k+1,\Omega^+}^T \mathbf{a}^{(k+1)} \Phi_{k+1,\Omega^+} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \right) \quad (55)$$

with

$$G_{k+1,\Omega^+}^0 = \left[-\Delta + \bar{\mu}_k + L^{-2} Q_{k+1,\Omega^+}^T \mathbf{a}^{(k+1)} Q_{k+1,\Omega^+} \right]_{\Omega_1}^{-1} \quad (56)$$

and at $\Phi_{k,\Omega_{k+1}} = \Psi_{k,\Omega_{k+1}}(\Omega^+)$ where

$$\begin{aligned} \Psi_{k,\Omega_{k+1}}(\Omega^+) &\equiv \Psi_{k,\Omega_{k+1}}(\Phi_{k+1}, \phi_{k+1,\Omega^+}^0) \\ &= Q_k \phi_{k+1,\Omega^+}^0 - \frac{aL^{-2}}{a_k + aL^{-2}} Q^T Q_{k+1} \phi_{k+1,\Omega^+}^0 + \frac{aL^{-2}}{a_k + aL^{-2}} Q^T \Phi_{k+1} \end{aligned} \quad (57)$$

2. Let Ψ_{k,Ω^+} be $\Phi_{k,\Omega}$ with $\Phi_{k,\Omega_{k+1}}$ replaced by the minimizer $\Psi_{k,\Omega_{k+1}}(\Omega^+)$, that is

$$\Psi_{k,\Omega^+} \equiv (\Phi_{1,\delta\Omega_1}, \dots, \Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}(\Omega^+)) \quad (58)$$

Then the minimizer in ϕ can also be written $\phi_{k,\Omega}(\phi_{\Omega_1^c}, \Psi_{k,\Omega^+})$ so we have the identity

$$\phi_{k+1,\Omega^+}^0 = \phi_{k,\Omega}(\phi_{\Omega_1^c}, \Psi_{k,\Omega^+}) \quad (59)$$

3. The value of $J_{\Omega^+}(\Phi_{k+1}, \Phi_{k,\Omega}, \phi)$ at the minimizer is:

$$\frac{1}{2} \left\langle \phi_{\Omega_1^c}, [-\Delta + \bar{\mu}_k]_{\Omega_1^c} \phi_{\Omega_1^c} \right\rangle + \left\langle \phi_{\Omega_1^c}, [-\Delta]_{\Omega_1, \Omega_1^c} \phi_{k+1,\Omega^+}^0 \right\rangle + S_{k+1}^0(\Omega_1, \Phi_{k+1,\Omega^+}, \phi_{k+1,\Omega^+}^0) \quad (60)$$

where

$$\begin{aligned} S_{k+1}^0(\Omega_1, \Phi_{k+1,\Omega^+}, \phi) &= \frac{1}{2} \sum_{j=1}^k a_j^{(k)} \|\Phi_j - Q_j \phi\|_{\delta\Omega_j}^2 + \frac{a_{k+1}}{2L^2} \|\Phi_{k+1} - Q_{k+1} \phi\|_{\Omega_{k+1}}^2 \\ &\quad + \frac{1}{2} \left\langle \phi, [-\Delta + \bar{\mu}_k]_{\Omega_1} \phi \right\rangle \end{aligned} \quad (61)$$

Proof. Recalling the expression (29) for $\|(\mathbf{a}^{(k)})^{\frac{1}{2}}(\Phi_{k,\Omega} - Q_{k,\Omega} \phi)\|^2$ we find that the variational equations for J in $\phi = \phi_{\Omega_1}$ and $\Phi_k = \Phi_{k,\Omega_{k+1}}$ are

$$\begin{aligned} \left(a_k + \frac{a}{L^2} Q^T Q \right) \Phi_k &= a_k Q_k \phi + Q^T \Phi_{k+1} \\ \left(-\Delta + \bar{\mu}_k + Q_{k,\Omega}^T \mathbf{a}^{(k)} Q_{k,\Omega} \right) \phi &= Q_{k,\Omega}^T \mathbf{a}^{(k)} \Phi_{k,\Omega} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \end{aligned} \quad (62)$$

Both of these we have seen before. The first equation is solved by $\Phi_k = \Psi_k(\Phi_{k+1}, \phi)$ defined in (34). We substitute this into the second equation. First note that splitting $\Phi_{k,\Omega_k} = \Phi_{k,\delta\Omega_k} + \Phi_{k,\delta\Omega_{k+1}}$ we have

$$Q_{k,\Omega}^T \mathbf{a}^{(k)} \Phi_{k,\Omega} = \sum_{j=1}^{k-1} a_j^{(k)} Q_j^T \Phi_{j,\delta\Omega_j} + a_k Q_k^T \Phi_{k,\Omega_{k+1}} \quad (63)$$

² Q_{k+1,Ω^+} is also written $Q_{\Omega^+, \mathbb{T}-k}$.

The substitution $\Phi_{k,\Omega_{k+1}} = \Psi_k$ goes in the last term here, and we have on Ω_{k+1}

$$a_k Q_k^T \Psi_k = a_k Q_k^T Q_k \phi - a_{k+1} L^{-2} Q_{k+1}^T Q_{k+1} \phi + a_{k+1} L^{-2} Q_{k+1}^T \Phi_{k+1} \quad (64)$$

Then the second equation becomes

$$\begin{aligned} & \left(-\Delta + \bar{\mu}_k + \sum_{j=1}^k [Q_j^T a_j^{(k)} Q_j]_{\delta\Omega_j} + L^{-2} [Q_{k+1}^T a_{k+1} Q_{k+1}]_{\Omega_{k+1}} \right) \phi \\ &= \sum_{j=1}^k a_j^{(k)} Q_j^T \Phi_{j,\delta\Omega_j} + L^{-2} a_{k+1} Q_{k+1}^T \Phi_{k+1} + [\Delta]_{\Omega_1, \Omega_1^c} \phi_{\Omega_1^c} \end{aligned} \quad (65)$$

This has the solution $\phi = \phi_{k+1, \Omega^+}^0$, and with this choice the first equation is solved by $\Phi_k = \Psi_k(\Phi_{k+1}, \phi_{k+1, \Omega^+}^0) \equiv \Psi_{k, \Omega_{k+1}}(\Omega^+)$. This establishes (55), (57).

Replace $\Phi_{k, \Omega}$ by Ψ_{k, Ω^+} in the second equation in (62) and solve for ϕ . We find that $\phi = \phi_{k, \Omega}(\phi_{\Omega_1^c}, \Psi_{k, \Omega^+})$. This gives the second representation for the minimizer and establishes (59).

To evaluate J at the minimum split the term $\frac{1}{2} \|\mathbf{a}^{1/2}(\Phi_{k, \Omega} - Q_{k, \Omega} \phi)\|_{\Omega_1}^2$ into a piece in Ω_{k+1} and a piece in $\Omega_1 - \Omega_{k+1}$. Then the value at the minimum is

$$\begin{aligned} J_{\Omega^+}(\Phi_{k+1}, \Psi_{k, \Omega^+}, (\phi_{\Omega_1^c}, \phi_{k+1, \Omega^+}^0)) &= \frac{a}{2L^2} \|\Phi_{k+1} - Q\Psi_k\|_{\Omega_{k+1}}^2 + \frac{a_k}{2} \|\Psi_k - Q_k \phi\|_{\Omega_{k+1}}^2 \\ &+ \frac{1}{2} \sum_{j=1}^k a_j^{(k)} \|\Phi_j - Q_j \phi\|_{\delta\Omega_j}^2 + \frac{1}{2} \left\langle \phi, (-\Delta + \bar{\mu}_k) \phi \right\rangle \quad \text{at} \quad \phi_{\Omega_1} = \phi_{k+1, \Omega^+}^0, \Psi_k = \Psi_{k, \Omega_{k+1}}(\Omega^+) \end{aligned} \quad (66)$$

However just as in (35) the first two terms combine to give $\frac{1}{2} a_{k+1} L^{-2} \|\Phi_{k+1} - Q_{k+1} \phi\|_{\Omega_{k+1}}^2$ so this is the same as

$$\frac{1}{2} \sum_{j=1}^k a_j^{(k)} \|\Phi_j - Q_j \phi\|_{\delta\Omega_j}^2 + \frac{a_{k+1}}{2L^2} \|\Phi_{k+1} - Q_{k+1} \phi\|_{\Omega_{k+1}}^2 + \frac{1}{2} \left\langle \phi, (-\Delta + \bar{\mu}_k) \phi \right\rangle \quad \text{at} \quad \phi_{\Omega_1} = \phi_{k+1, \Omega^+}^0 \quad (67)$$

This is the same as (60) and this completes the proof.

Remarks. We develop some consequences of these results. Suppose we expand around the minimizer for $J_{\Omega^+}(\Phi_{k+1}, \Phi_{k, \Omega}, (\phi_{\Omega_1^c}, \phi_{k, \Omega}))$ in $\Phi_{k, \Omega_{k+1}}$, namely $\Psi_{k, \Omega_{k+1}}(\Omega^+)$. Put $\Phi_{k, \Omega_{k+1}} = \Psi_{k, \Omega_{k+1}}(\Omega^+) + Z$ and hence $\Phi_{k, \Omega} = \Psi_{k, \Omega^+} + (0, Z)$. Then $\phi_{k, \Omega} = \phi_{k, \Omega}(\phi_{\Omega_1^c}, \Phi_{k, \Omega})$ becomes

$$\phi_{k, \Omega}(\phi_{\Omega_1^c}, \Psi_{k, \Omega^+} + (0, Z)) = \phi_{k, \Omega}(\phi_{\Omega_1^c}, \Psi_{k, \Omega^+}) + a_k G_{k, \Omega} Q_k^T Z = \phi_{k+1, \Omega^+}^0 + \mathcal{Z}_{k, \Omega} \quad (68)$$

Here we have used (59) and defined $\mathcal{Z}_{k, \Omega} = \phi_{k, \Omega}(0, Z) = a_k G_{k, \Omega} Q_k^T Z$. Now we claim that

$$\begin{aligned} & J_{\Omega^+}(\Phi_{k+1}, \Psi_{k, \Omega^+} + (0, Z), (\phi_{\Omega_1^c}, \phi_{k+1, \Omega^+}^0 + \mathcal{Z}_{k, \Omega})) \\ &= \frac{1}{2} \left\langle \phi_{\Omega_1^c}, [-\Delta + \bar{\mu}_k]_{\Omega_1^c} \phi_{\Omega_1^c} \right\rangle + \left\langle \phi_{\Omega_1^c}, [-\Delta]_{\Omega_1, \Omega_1^c} \phi_{k+1, \Omega^+}^0 \right\rangle + S_{k+1}^0(\Omega_1, \Phi_{k+1, \Omega^+}, \phi_{k+1, \Omega^+}^0) \\ &+ \frac{1}{2} \left(Z, \left[\Delta_{k, \Omega} + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} Z \right) \end{aligned} \quad (69)$$

That the first three terms are the value at $Z = 0$ follows from the previous lemma. The linear terms must vanish. Thus we only have to look at the quadratic terms in Z which are

$$\frac{a}{2L^2} \|QZ\|_{\Omega_{k+1}}^2 + S_k(\Omega_1, (0, Z), \mathcal{Z}_{k, \Omega}) = \frac{a}{2L^2} \|QZ\|_{\Omega_{k+1}}^2 + \frac{1}{2} \left(Z, \left[\Delta_{k, \Omega} \right]_{\Omega_{k+1}} Z \right) \quad (70)$$

The second form follows from (52). Hence (69) is established.

The original integral (53) with Z as the integration variable would now be evaluated as

$$\begin{aligned} & \exp \left(-\frac{1}{2} \left\langle \phi_{\Omega_1^c}, [-\Delta + \bar{\mu}_k]_{\Omega_1^c} \phi_{\Omega_1^c} \right\rangle - \left\langle \phi_{\Omega_1^c}, [-\Delta]_{\Omega_1, \Omega_1^c} \phi_{k+1, \Omega^+}^0 \right\rangle - S_{k+1}^0(\Omega_1, \Phi_{k+1, \Omega^+}, \phi_{k+1, \Omega^+}^0) \right) \\ & \int \exp \left(-\frac{1}{2} \left(Z, \left[\Delta_k, \Omega + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} Z \right) dZ \end{aligned} \quad (71)$$

Let us also check that this scales the way we expect. As in section 2.1 we replace each Ω_j by $L\Omega_j$ and each field like $\Phi_{j, \delta\Omega_j}$ by $[\Phi_{j, L}]_{L\delta\Omega_j} = [\Phi_{j, \delta\Omega_j}]_L$. Since $\bar{\mu}_k = L^{-2}\bar{\mu}_{k+1}$ and $a_j^{(k)} = L^{-2}a_j^{(k+1)}$ and Q_j is scale invariant we have

$$\begin{aligned} & \left(-\Delta + \bar{\mu}_k + L^{-2} Q_{k+1, \Omega^+}^T \mathbf{a}^{(k+1)} Q_{k+1, \Omega^+} \right) f_L \\ & = L^{-2} \left((-\Delta + \bar{\mu}_{k+1} + Q_{k+1, \Omega^+}^T \mathbf{a}^{(k+1)} Q_{k+1, \Omega^+}) f \right)_L \end{aligned} \quad (72)$$

It follows that

$$G_{k, L\Omega^+}^0 f_L = L^2 [G_{k, \Omega^+} f]_L \quad (73)$$

and hence from (55)

$$\phi_{k+1, L\Omega^+}^0 \left([\phi_{\Omega_1^c}]_L, [\Phi_{k+1, \Omega^+}]_L \right) = \left[\phi_{k+1, \Omega^+}(\phi_{\Omega_1^c}, \Phi_{k+1, \Omega^+}) \right]_L \quad (74)$$

Then

$$S_{k+1}^0 \left(L\Omega_1, [\Phi_{k+1, \Omega^+}]_L, [\phi_{k+1, \Omega^+}]_L \right) = S_{k+1} \left(\Omega_1, \Phi_{k+1, \Omega^+}, \phi_{k+1, \Omega^+} \right) \quad (75)$$

The other terms in the exponent in (71) scale similarly, and thus they become

$$\exp \left(-\frac{1}{2} \left\langle \phi_{\Omega_1^c}, [-\Delta + \bar{\mu}_{k+1}]_{\Omega_1^c} \phi_{\Omega_1^c} \right\rangle - \left\langle \phi_{\Omega_1^c}, [-\Delta]_{\Omega_1, \Omega_1^c} \phi_{k+1, \Omega^+} \right\rangle - S_{k+1} \left(\Omega_1, \Phi_{k+1, \Omega^+}, \phi_{k+1, \Omega^+} \right) \right) \quad (76)$$

as expected.

2.4 a variation

We actually use a variation of the previous section. First suppose Λ is any union of M -cubes in \mathbb{T}_{M+N-k}^{-k} and define

$$S_k^*(\Lambda, \Phi_k, \Omega, \phi) = \frac{1}{2} \|\mathbf{a}^{1/2}(\Phi_k, \Omega - Q_k, \Omega \phi)\|_\Lambda^2 + \frac{1}{2} \|\partial\phi\|_{*, \Lambda}^2 + \frac{1}{2} \bar{\mu}_k \|\phi\|_\Lambda^2 \quad (77)$$

Here $\|\partial\phi\|_{*, \Lambda}^2$ contains half the bonds that cross the boundary of Λ . Precisely it is defined for $\phi : \mathbb{T}_{M+N-k}^{-k} \rightarrow \mathbb{R}$ by

$$\|\partial\phi\|_{*, \Lambda}^2 = \sum_{\langle x, x' \rangle \in \Lambda} L^{-3k} |\partial\phi(x, x')|^2 + \frac{1}{2} \sum_{x \in \Lambda, x' \in \Lambda^c} L^{-3k} |\partial\phi(x, x')|^2 \quad (78)$$

This has the advantage that if Λ_1, Λ_2 are disjoint (but possibly with a common boundary), then

$$\|\partial\phi\|_{*, \Lambda_1 \cup \Lambda_2}^2 = \|\partial\phi\|_{*, \Lambda_1}^2 + \|\partial\phi\|_{*, \Lambda_2}^2 \quad (79)$$

A similar decomposition holds for $S_k^*(\Lambda, \Phi_k, \Omega, \phi)$.

Also suppose the free action is in a set Λ smaller than Ω_1 . In fact suppose we have

$$\Omega_1 \supset \Omega_2 \supset \dots \supset \Omega_k \supset \Lambda \supset \Omega_{k+1} \quad (80)$$

with separation between Ω_1^c and Λ . In that case (77) becomes

$$S_k^*(\Lambda, \Phi_k, \phi) = \frac{a_k}{2} \|\Phi_k - Q_k \phi\|_\Lambda^2 + \frac{1}{2} \|\partial \phi\|_{*,\Lambda}^2 + \frac{1}{2} \bar{\mu}_k \|\phi\|_\Lambda^2 \quad (81)$$

We study what happens to this expression if we make expansions around the minimizer for the original problem $\phi = \phi_{k,\Omega} = \phi_{k,\Omega}(\phi_{\Omega^c}, \Phi_{k,\Omega})$ (which satisfies useful identities). Although this is not the minimizer for the current problem we will obtain a similar result.

Lemma 2.4. For $Z : \mathbb{T}_{M+N-k}^0 \rightarrow \mathbb{R}$ and $\mathcal{Z} : \mathbb{T}_{M+N-k}^{-k} \rightarrow \mathbb{R}$ each defined on a neighborhood of Λ

$$\begin{aligned} S_k^*(\Lambda, \Phi_k + Z, \phi_{k,\Omega} + \mathcal{Z}) &= S_k^*(\Lambda, \Phi_k, \phi_{k,\Omega}) + S_k^*(\Lambda, Z, \mathcal{Z}) \\ &\quad + a_k \langle Z, (\Phi_k - Q_k \phi_{k,\Omega}) \rangle_\Lambda + \mathbf{b}_\Lambda(\partial \phi_{k,\Omega}, \mathcal{Z}) \end{aligned} \quad (82)$$

where the boundary term is

$$\mathbf{b}_\Lambda(\partial \phi, \mathcal{Z}) \equiv \frac{1}{2} \sum_{x \in \Lambda, x' \in \Lambda^c} L^{-2k} \partial \phi(x, x') (\mathcal{Z}(x) + \mathcal{Z}(x')) \quad (83)$$

Proof. Every thing is quadratic so it suffices to identify cross terms. These are

$$\begin{aligned} a_k \langle Z, (\Phi_k - Q_k \phi_{k,\Omega}) \rangle_\Lambda \\ -a_k \langle (\Phi_k - Q_k \phi_{k,\Omega}), Q_k \mathcal{Z} \rangle_\Lambda + \langle \partial \phi_{k,\Omega}, \partial \mathcal{Z} \rangle_{*,\Lambda} + \bar{\mu}_k \langle \phi_{k,\Omega}, \mathcal{Z} \rangle_\Lambda \end{aligned} \quad (84)$$

In appendix B it is shown that

$$\langle \partial \phi_{k,\Omega}, \partial \mathcal{Z} \rangle_{*,\Lambda} = \langle (-\Delta) \phi_{k,\Omega}, \mathcal{Z} \rangle_\Lambda + \mathbf{b}_\Lambda(\partial \phi_{k,\Omega}, \mathcal{Z}) \quad (85)$$

Then our expression becomes

$$\begin{aligned} a_k \langle Z, (\Phi_k - Q_k \phi_{k,\Omega}) \rangle_\Lambda \\ -a_k \langle Q_k^T \Phi_k, \mathcal{Z} \rangle_\Lambda + \left\langle (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k) \phi_{k,\Omega}, \mathcal{Z} \right\rangle_\Lambda + \mathbf{b}_\Lambda(\partial \phi_{k,\Omega}, \mathcal{Z}) \end{aligned} \quad (86)$$

But the second and third terms combine to zero by the definition (42) of $\phi_{k,\Omega}$. There is no contribution from $\phi_{\Omega_1^c}$ due to our separation assumption. This completes the proof.

Continuing with our assumption on Λ , we now investigate how the single step analysis of section 2.3 changes, particularly (69). We introduce

$$J_{\Lambda, \Omega_{k+1}}^*(\Phi_{k+1}, \Phi_k, \phi) = \frac{a}{2L^2} \|\Phi_{k+1} - Q \Phi_k\|_{\Omega_{k+1}}^2 + S_k^*(\Lambda, \Phi_k, \phi) \quad (87)$$

and expand in $\Phi_{k,\Omega_{k+1}}$ and ϕ around the minima $\Psi_{k,\Omega_{k+1}}(\Omega^+)$ and $\phi_{k+1,\Omega^+}^0 = \phi_{k,\Omega}(\phi_{\Omega_1^c}, \Psi_{k,\Omega^+})$ for the original problem with Ω^+ . The result is the following

Lemma 2.5. For $Z : \Omega_{k+1}^{(k)} \rightarrow \mathbb{R}$ and $\mathcal{Z}_{k,\Omega} = \phi_{k,\Omega}(0, Z)$

$$\begin{aligned} J_{\Lambda, \Omega_{k+1}}^* \left(\Phi_{k+1}, \Psi_{k,\Omega^+} + (0, Z), \phi_{k+1,\Omega^+}^0 + \mathcal{Z}_{k,\Omega} \right) \\ = S_{k+1}^{*,0}(\Lambda, \Phi_{k+1,\Omega^+}, \phi_{k+1,\Omega^+}^0) + \frac{1}{2} \left\langle Z, \left[\Delta_{k,\Omega} + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} Z \right\rangle + R_{k,\Omega,\Lambda} + \mathbf{b}_\Lambda(\partial \phi_{k,\Omega}, \mathcal{Z}) \end{aligned} \quad (88)$$

where

$$\begin{aligned} S_{k+1}^{*,0}(\Lambda, \Phi_{k+1}, \Omega^+, \phi) &= \frac{a_{k+1}}{2L^2} \|\Phi_{k+1} - Q_{k+1}\phi\|_{\Omega_{k+1}}^2 + \frac{a_k}{2} \|\Phi_k - Q_k\phi\|_{\Lambda - \Omega_{k+1}}^2 \\ &\quad + \frac{1}{2} \|\partial\phi\|_{*,\Lambda}^2 + \frac{1}{2} \bar{\mu}_k \|\phi\|_{\Lambda}^2 \end{aligned} \quad (89)$$

and

$$R_{k,\Omega,\Lambda} \equiv -\frac{1}{2} \|\mathbf{a}^{1/2} Q_{k,\Omega} \mathcal{Z}_{k,\Omega}\|_{\Lambda^c}^2 + \|\partial \mathcal{Z}_{k,\Omega}\|_{*,\Lambda}^2 + \frac{1}{2} \bar{\mu}_k \|\mathcal{Z}_{k,\Omega}\|_{\Lambda^c}^2 \quad (90)$$

Remark. $S_{k+1}^{*,0}(\Lambda, \Phi_{k+1}, \Omega^+, \phi_{k+1}^0, \Omega^+)$ scales to $S_{k+1}^*(\Lambda, \Phi_{k+1}, \Omega^+, \phi_{k+1}, \Omega^+)$ as in (75).

Proof. By the previous lemma our expression is

$$\begin{aligned} &\frac{a}{2L^2} \|\Phi_{k+1} - Q\Psi_{k,\Omega_{k+1}}(\Omega^+)\|_{\Omega_{k+1}}^2 + \frac{a}{2L^2} \|QZ\|_{\Omega_{k+1}}^2 - \frac{a}{L^2} < QZ, (\Phi_{k+1} - Q\Psi_{k,\Omega_{k+1}}(\Omega^+)) > \\ &+ S_k^*(\Lambda, \Psi_{k,\Omega_{k+1}}(\Omega^+), \phi_{k+1}^0, \Omega^+) + S_k^*(\Lambda, (0, Z), \mathcal{Z}_{k,\Omega}) \\ &+ a_k < Z, (\Psi_{k,\Omega_{k+1}}(\Omega^+) - Q_k\phi_{k+1}^0, \Omega^+) >_{\Omega_{k+1}} + \mathbf{b}_{\Lambda}(\partial\phi_{k,\Omega}, \mathcal{Z}) \end{aligned} \quad (91)$$

However the linear terms vanish since

$$\frac{a}{L^2} Q^T(\Phi_{k+1} - Q\Psi_{k,\Omega_{k+1}}(\Omega^+)) = \frac{a_{k+1}}{L^2} Q^T(\Phi_{k+1} - Q_{k+1}\phi_{k+1}^0, \Omega^+) = a_k(\Psi_{k,\Omega_{k+1}}(\Omega^+) - Q_k\phi_{k+1}^0, \Omega^+) \quad (92)$$

as one can check by inserting the definition of $\Psi_{k,\Omega_{k+1}}(\Omega^+)$ from (57). Also as in (66), (67)

$$\frac{a}{2L^2} \|\Phi_{k+1} - Q\Psi_{k,\Omega_{k+1}}(\Omega^+)\|_{\Omega_{k+1}}^2 + S_k^*(\Lambda, \Psi_{k,\Omega_{k+1}}(\Omega^+), \phi_{k+1}^0, \Omega^+) = S_{k+1}^{*,0}(\Lambda, \Phi_{k+1}, \Omega^+, \phi_{k+1}^0, \Omega^+) \quad (93)$$

Finally

$$\begin{aligned} \frac{a}{2L^2} \|QZ\|_{\Omega_{k+1}}^2 + S_k^*(\Lambda, (0, Z), \mathcal{Z}_{k,\Omega}) &= \frac{a}{2L^2} \|QZ\|_{\Omega_{k+1}}^2 + S_k(\Omega_1, (0, Z), \mathcal{Z}_{k,\Omega}) + R_{k,\Omega,\Lambda} \\ &= \frac{1}{2} \left\langle Z, \left[\Delta_{k,\Omega} + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} Z \right\rangle + R_{k,\Omega,\Lambda} \end{aligned} \quad (94)$$

The last step is from (70). This completes the proof.

2.5 random walk expansion

We analyze the propagator

$$G_{k,\Omega} = \left[-\Delta + \bar{\mu}_k + Q_{k,\Omega}^T \mathbf{a} Q_{k,\Omega} \right]_{\Omega_1}^{-1} \quad (95)$$

with Dirichlet boundary conditions, defined on functions on $\Omega_1 \subset \mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k}$. We will need a random walk expansion for this operator analogous to the global expansion explained in part I.

Recall that $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_k)$ with $\Omega_j \supset \Omega_{j+1}$ and Ω_j a union of $L^{-(k-j)}M$ cubes. For the random walk expansion we impose the separation condition

$$d(\Omega_j^c, \Omega_{j+1}) \geq L^{-(k-j)}MR \quad (96)$$

for some positive integer $R = \mathcal{O}(1)$. Hence $\delta\Omega_j = \Omega_j - \Omega_{j+1}$ has a minimum width $L^{-(k-j)}MR$. There is an associated scaled distance that will play a role in what follows. It is defined for $(x, y) \in \cup_{j=1}^k \delta\Omega_j^{(j)}$ by

$$d_{\Omega}(x, y) = \inf_{\gamma: x \rightarrow y} \sum_{j=1}^k L^{k-j} \ell(\gamma \cap \delta\Omega_j) \quad (97)$$

with $\delta\Omega_k = \Omega_k$. Here γ is a path joining x, y in the lattice $\mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k}$ such that in $\delta\Omega_j$ the path γ consists of $L^{-(k-j)}$ links in $\delta\Omega_j^{(j)}$. The factor L^{k-j} in $d_{\mathbf{\Omega}}(x, y)$ means we count these links as unit length. For $0 \leq \delta \leq 1$ if MR is sufficiently large this satisfies the bound (Lemma 2.1 in [6])

$$\sum_y \exp\left(-\delta d_{\mathbf{\Omega}}(x, y)\right) \leq \mathcal{O}(1)\delta^{-3} \quad (98)$$

Now $\delta\Omega_j$ is partitioned into $L^{-(k-j)}M = L^{-k+j+m}$ cubes \square_z centered on the points $z \in \delta\Omega_j^{(j+m)}$. Correspondingly there is a multiscale partition \square_z of $\Omega_1 = \delta\Omega_1 \cup \delta\Omega_2 \cup \dots \cup \delta\Omega_{k-1} \cup \Omega_k$. We also consider enlargements $\tilde{\square}_z$ which are centered on the same points but have width $3L^{-(k-j)}M$ in $\delta\Omega_j$. They provide a cover of Ω_1 .

The random walk expansion is based on local inverses based on the cubes $\tilde{\square} = \tilde{\square}_z$ we define

$$G_{k,\mathbf{\Omega}}(\tilde{\square}) = \left[-\Delta + \bar{\mu}_k + Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}} \right]_{\tilde{\square} \cap \Omega_1}^{-1} \quad (99)$$

Here $[-\Delta]_{\tilde{\square} \cap \Omega_1}$ is taken with Neumann boundary conditions on the part of the boundary of $\tilde{\square} \cap \Omega_1$ in Ω_1 , and Dirichlet boundary conditions of the part of the boundary shared with $\partial\Omega_1$. Away from $\partial\Omega_1$ it is just $[-\Delta]_{\tilde{\square}}$ with Neumann conditions, as in part I.

Define

$$\Delta_y = L^{-(k-j)} \text{ cubes centered on } y \in \delta\Omega_j^{(j)} \quad (100)$$

These give a finer partition of $\delta\Omega_j$ and hence also a partition of Ω_1 . A basic result is the following:

Lemma 2.6. *Let $\Delta_y \subset \tilde{\square} \cap \delta\Omega_j$ and $\Delta_{y'} \subset \tilde{\square} \cap \delta\Omega_{j'}$, $|j - j'| \leq 1$. Then with $\gamma_0 = \mathcal{O}(L^{-2})$*

$$\begin{aligned} |1_{\Delta_y} G_{k,\mathbf{\Omega}}(\tilde{\square}) 1_{\Delta_{y'}} f| &\leq CL^{-2(k-j)} e^{-\frac{1}{2}\gamma_0 d_{\mathbf{\Omega}}(y, y')} \|f\|_{\infty} \\ |1_{\Delta_y} \partial G_{k,\mathbf{\Omega}}(\tilde{\square}) 1_{\Delta_{y'}} f| &\leq CL^{-(k-j)} e^{-\frac{1}{2}\gamma_0 d_{\mathbf{\Omega}}(y, y')} \|f\|_{\infty} \\ |1_{\Delta_y} \delta_{\alpha} \partial G_{k,\mathbf{\Omega}}(\tilde{\square}) 1_{\Delta_{y'}} f| &\leq CL^{-(1-\alpha)(k-j)} e^{-\frac{1}{2}\gamma_0 d_{\mathbf{\Omega}}(y, y')} \|f\|_{\infty} \end{aligned} \quad (101)$$

Remark.

1. Here δ_{α} is the Holder derivative defined for $x \neq x'$ by $(\delta_{\alpha} f)(x, x') = (f(x) - f(x'))d(x, x')^{-\alpha}$. We take $\frac{1}{2} < \alpha < 1$.
2. There is another way to state this bound which will be useful. It is

$$\begin{aligned} &\left| 1_{\Delta_y} G_{k,\mathbf{\Omega}}(\tilde{\square}) 1_{\Delta_{y'}} f \right|, \quad L^{-(k-j)} \left| 1_{\Delta_y} \partial G_{k,\mathbf{\Omega}}(\tilde{\square}) 1_{\Delta_{y'}} f \right|, \quad L^{-(1+\alpha)(k-j)} \left| 1_{\Delta_y} \delta_{\alpha} \partial G_{k,\mathbf{\Omega}}(\tilde{\square}) 1_{\Delta_{y'}} f \right| \\ &\leq CL^{-2(k-j')} e^{-\frac{1}{2}\gamma_0 d_{\mathbf{\Omega}}(y, y')} \|f\|_{\infty} \end{aligned} \quad (102)$$

Since $|j - j'| \leq 1$ this follows from (101).

Proof. First suppose that $\Delta_y, \Delta_{y'} \subset \tilde{\square} \subset \delta\Omega_j$. In this circumstance we have $\tilde{\square} \cap \Omega_1 = \tilde{\square}$ and

$$G_{k,\mathbf{\Omega}}(\tilde{\square}) = [-\Delta + \bar{\mu}_k + a_j L^{2(k-j)} Q_j^T Q_j]_{\tilde{\square}}^{-1} \quad (103)$$

We need to prove the bound with $d_{\mathbf{\Omega}}(y, y') = L^{k-j}d(y, y')$. We scale up the the required estimate from $\mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k}$ to $\mathbb{T}_{\mathbf{M}+\mathbf{N}-j}^{-j}$ and prove it there. Replace f by $f_{L^{-(k-j)}}$ where $f : \mathbb{T}_{\mathbf{M}+\mathbf{N}-j}^{-j} \rightarrow \mathbb{R}$, and

replace $\tilde{\square}$ by $L^{-(k-j)}\tilde{\square}$ where now $\tilde{\square}$ is now a $3M$ -cube in \mathbb{T}_{M+N-j}^{-j} . Then $G_{k,\Omega}(L^{-(k-j)}\tilde{\square})f_{L^{-(k-j)}} = L^{-2(k-j)}[G_j(\tilde{\square})f]_{L^{-(k-j)}}$ where

$$G_k(\tilde{\square}) = [-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k]_{\tilde{\square}}^{-1} \quad (104)$$

is the standard propagator on \mathbb{T}_{M+N-k}^{-k} . For the bounds (101) it now suffices to prove that for unit cubes $\Delta_y, \Delta_{y'}$ and $x, x' \in \Delta_y$, $\text{supp} f \subset \Delta_{y'}$:

$$|(G_k(\tilde{\square})f)(x)|, |(\partial G_k(\tilde{\square})f)(x)|, |(\delta_\alpha \partial G_k(\tilde{\square})f)(x, x')| \leq C e^{-\gamma_0 d(y, y')} \|f\|_\infty \quad (105)$$

These are already established; see [4] or Appendix D in part I.

It may happen that $\tilde{\square}$ is not in a single $\delta\Omega_j$. Suppose that $\tilde{\square}$ intersects both $\delta\Omega_j$ and $\delta\Omega_{j+1}$. Then

$$G_{k,\Omega}(\tilde{\square}) = \left[-\Delta_{\tilde{\square}} + \bar{\mu}_k + a_j L^{2(k-j)} [Q_j^T Q_j]_{\tilde{\square} \cap \delta\Omega_j} + a_{j+1} L^{2(k-(j+1))} [Q_{j+1}^T Q_{j+1}]_{\tilde{\square} \cap \delta\Omega_{j+1}} \right]^{-1} \quad (106)$$

Since $d_\Omega(y, y') \leq L^{k-j} d(y, y')$ it again suffices to prove (101) with the $L^{k-j} d(y, y')$ in the exponential. Again we scale up to \mathbb{T}_{M+N-j}^{-j} where the propagator becomes

$$G'_j(\tilde{\square}) \equiv \left[-\Delta_{\tilde{\square}} + \bar{\mu}_j + a_j [Q_j^T Q_j]_{\tilde{\square} \cap \delta\Omega_j} + \frac{a_{j+1}}{L^2} [Q_{j+1}^T Q_{j+1}]_{\tilde{\square} \cap \delta\Omega_{j+1}} \right]^{-1} \quad (107)$$

We must establish that $G'_j(\tilde{\square})$ satisfies bounds of the form (105), now with Δ_y a unit cube, $\Delta_{y'}$ an L -cube, and $x, x' \in \Delta_y$, $\text{supp} f \subset \Delta_{y'}$: In this case we use the identity ([6], p. 230)

$$G'_j(\tilde{\square}) = G_j(\tilde{\square}) + a_j^2 G_j(\tilde{\square}) Q_j^T C_j(\tilde{\square} \cap \delta\Omega_{j+1}) Q_j G_j(\tilde{\square}) \quad (108)$$

Here all the pieces have pointwise bounds of the form we want and this yields the the result. For $G_j(\tilde{\square})$ use (105) and for $C_j(\tilde{\square} \cap \delta\Omega_j)$ see [4] or Appendix D in part I. Also note the following piece of the estimate. If the L -cube $\Delta_{y'}$ is written as a union of unit cubes $\Delta_{y''}$ then for $x \in \Delta_y$, $\text{supp} f \subset \Delta'_{y'}$

$$|(G_j(\tilde{\square})f)(x)| \leq \sum_{y''} |(G_j(\tilde{\square})1_{\Delta_{y''}}f)(x)| \leq C \sum_{y''} e^{-\gamma_0 d(y, y'')} \|f\|_\infty \leq C e^{-\frac{1}{2}\gamma_0 d(y, y')} \|f\|_\infty \quad (109)$$

Here we used $d(y, y'') \geq d(y, y') - L$

There is another special case that needs to be considered, namely when $\tilde{\square}$ touches or intersects Ω_1^c . In this case $\tilde{\square} \cap \Omega_1$ may not be rectangular and we may have mixed boundary conditions and so the pointwise bounds (105) may not hold. However we do still have L^2 bounds even for non-rectangular regions and mixed boundary conditions. In the scaled version instead of (104) we have for a $3M$ cube $\tilde{\square}$ in \mathbb{T}_{M+N-1}^{-1}

$$G_1(\tilde{\square}) = [-\Delta + \bar{\mu}_1 + a_1 Q^T Q]_{\tilde{\square} \cap \Omega_1}^{-1} \quad (110)$$

Instead of (105) we have for unit cubes $\Delta_y, \Delta'_y \subset \tilde{\square} \cap \Omega_1$

$$\|1_{\Delta_y} G_1(\tilde{\square}) 1_{\Delta'_y} f\|_2 \leq C e^{-\gamma_0 d(y, y')} \|f\|_2 \quad (111)$$

However on this L^{-1} lattice we have $L^{-3} \|f\|_{\infty, \Delta_y} \leq \|f\|_{2, \Delta_y} \leq \|f\|_{\infty, \Delta_y}$ so this implies

$$|1_{\Delta_y} G_1(\tilde{\square}) 1_{\Delta'_y} f| \leq C e^{-\gamma_0 d(y, y')} \|f\|_\infty \quad (112)$$

Also on this L^{-1} lattice this implies bounds on the derivatives. Thus $G_1(\tilde{\square})$ satisfies the bounds (105). This completes the proof.

A random walk or path is a sequence of points

$$\omega = (\omega_0, \omega_1, \dots, \omega_n) \quad (113)$$

in $\delta\Omega_1^{(1+m)} \cup \dots \cup \delta\Omega_{k-1}^{(k-1+m)} \cup \Omega_k^{(k+m)}$. These are the centers of the cubes in the multi scale partition $\{\square_z\}$. Successive points in the walk are required to be neighbors in the sense that the larger cubes satisfy $\tilde{\square}_{\omega_j} \cap \tilde{\square}_{\omega_{j+1}} \neq \emptyset$.

Theorem 2.2. *The Green's function $G_{k,\Omega}$ defined in (43) has a random walk expansion of the form*

$$G_{k,\Omega} = \sum_{\omega} G_{k,\Omega,\omega} \quad (114)$$

convergent for M sufficiently large. It yields the bounds for $\Delta_y \subset \delta\Omega_j$ and $\Delta_{y'} \subset \delta\Omega_{j'}$ as in (100):

$$\begin{aligned} |1_{\Delta_y} G_{k,\Omega} 1_{\Delta_{y'}} f| &\leq CL^{-2(k-j')} e^{-\frac{1}{4}\gamma_0 d_{\Omega}(y,y')} \|f\|_{\infty} \\ L^{-(k-j)} |1_{\Delta_y} \partial G_{k,\Omega} 1_{\Delta_{y'}} f| &\leq CL^{-2(k-j')} e^{-\frac{1}{4}\gamma_0 d_{\Omega}(y,y')} \|f\|_{\infty} \\ L^{-(1+\alpha)(k-j)} |1_{\Delta_y} \delta_{\alpha} \partial G_{k,\Omega} 1_{\Delta_{y'}} f| &\leq CL^{-2(k-j')} e^{-\frac{1}{4}\gamma_0 d_{\Omega}(y,y')} \|f\|_{\infty} \end{aligned} \quad (115)$$

Proof. We sketch the proof (see [4], [6], [10]). Let $0 \leq h_z \leq 1$ be such that h_z^2 be a smooth partition of unity subordinate to the covering $\{\tilde{\square}_z\}$ of Ω_1 . Thus $\text{supp } h_z \subset \tilde{\square}_z$ and $\sum_z h_z^2 = 1$ on a neighborhood of Ω_1 . Taking advantage of the size of $\tilde{\square}_z$ we can arrange that in $\delta\Omega_j$

$$|\partial h_z| \leq \mathcal{O}(1)(L^{-(k-j)} M)^{-1} \quad |\partial \partial h_z| \leq \mathcal{O}(1)(L^{-(k-j)} M)^{-2} \quad (116)$$

Define the parametrix

$$G_{k,\Omega}^* = \sum_z h_z G_{k,\Omega}(\tilde{\square}_z) h_z \quad (117)$$

Then

$$\begin{aligned} \left[-\Delta + \bar{\mu}_k + Q_{k,\Omega}^T \mathbf{a} Q_{k,\Omega} \right]_{\Omega_1} G_{k,\Omega}^* &= \sum_z h_z \left[-\Delta + \bar{\mu}_k + Q_{k,\Omega}^T \mathbf{a} Q_{k,\Omega} \right]_{\Omega_1} G_{k,\Omega}(\tilde{\square}_z) h_z \\ &\quad + \sum_z K_z G_{k,\Omega}(\tilde{\square}_z) h_z \end{aligned} \quad (118)$$

where

$$K_z = \left[\left[-\Delta + \bar{\mu}_k + Q_{k,\Omega}^T \mathbf{a} Q_{k,\Omega} \right]_{\Omega_1}, h_z \right] \quad (119)$$

In the first term, since $\text{supp } h_z$ is well inside $\tilde{\square}_z$ we impose add Neumann boundary conditions on $\tilde{\square}_z \cap \Omega_1$ with no change and identify $[-\Delta + \bar{\mu}_k + Q_{k,\Omega}^T \mathbf{a} Q_{k,\Omega}]_{\tilde{\square}_z \cap \Omega_1}$. This removes the operator $G_{k,\Omega}(\tilde{\square}_z)$ and leaves us with $\sum_z h_z^2 = 1$ Thus we have

$$\left[-\Delta + \bar{\mu}_k + Q_{k,\Omega}^T \mathbf{a} Q_{k,\Omega} \right]_{\Omega_1} G_{k,\Omega}^* = I - \sum_z R_z \equiv I - R \quad (120)$$

where

$$R_z = K_z G_{k,\Omega}(\tilde{\square}_z) h_z \quad (121)$$

Then if the series converges

$$\begin{aligned}
G_{k,\mathbf{\Omega}} &= G_{k,\mathbf{\Omega}}^* (I - R)^{-1} = G_{k,\mathbf{\Omega}}^* \sum_{n=0}^{\infty} R^n \\
&= \sum_{n=0}^{\infty} \sum_{\omega_0, \omega_1, \dots, \omega_n} \left(h_{\omega_0} G(\tilde{\square}_{\omega_0}) h_{\omega_0} \right) R_{\omega_1} \cdots R_{\omega_n} \\
&\equiv \sum_{\omega} G_{k,\mathbf{\Omega},\omega}
\end{aligned} \tag{122}$$

The last line defines $G_{k,\mathbf{\Omega},\omega}$. We have used that $R_z R_{z'} = 0$ unless $\tilde{\square}_z \cap \tilde{\square}_{z'} \neq \emptyset$ to identify the sum over walks.

We estimate $K_z f$. First for $x \in \Delta_y \subset \delta\Omega_j$ we have

$$|(-\Delta, h_z]f)(x)| \leq \mathcal{O}(1) \left((L^{-(k-j)} M)^{-2} \|1_{\Delta_y} f\|_{\infty} + (L^{-(k-j)} M)^{-1} \|1_{\Delta_y} \partial f\|_{\infty} \right) \tag{123}$$

Indeed the term $[-\Delta, h_z]$ is local and involves derivatives of h_z , so we get the indicated factors. The term $\left[Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}}, h_z \right] f$ can also be expressed in term of derivatives of h_z since it can be written in $\delta\Omega_j$ as

$$a_j^{(k)} \left(\left[Q_j^T Q_j, h_z \right] f \right)(x) = a_j L^{2(k-j)} L^{-3j} \sum_{x' \in B_j(x)} (h_z(x') - h_z(x)) f(x') \tag{124}$$

and so is estimated by

$$a_j^{(k)} \left| \left(\left[Q_j^T Q_j, h_z \right] f \right)(x) \right| \leq \mathcal{O}(1) L^{2(k-j)} M^{-1} \|1_{\Delta_y} f\|_{\infty} \tag{125}$$

Combining these we have for $x \in \Delta_y \subset \delta\Omega_j$

$$|(K_z f)(x)| \leq \mathcal{O}(1) M^{-1} \left(L^{2(k-j)} \|1_{\Delta_y} f\|_{\infty} + L^{(k-j)} \|1_{\Delta_y} \partial f\|_{\infty} \right) \tag{126}$$

Combining this bound with the bound (102) on $G_{k,\mathbf{\Omega}}(\tilde{\square}_z)$ yields for $x \in \Delta_y \subset \tilde{\square}_z \cap \delta\Omega_j$ and $\text{supp} f \subset \Delta_{y'} \subset \tilde{\square}_z \cap \delta\Omega_{j'}$:

$$\begin{aligned}
&L^{-2(k-j)} |(K_z G_{k,\mathbf{\Omega}}(\tilde{\square}_z) f)(x)| \\
&\leq \mathcal{O}(1) M^{-1} \left(\|1_{\Delta_y} G_{k,\mathbf{\Omega}}(\tilde{\square}_z) f\|_{\infty} + L^{-(k-j)} \|1_{\Delta_y} \partial G_{k,\mathbf{\Omega}}(\tilde{\square}_z) f\|_{\infty} \right) \\
&\leq C M^{-1} L^{-2(k-j')} e^{-\frac{1}{2}\gamma_0 d} \mathbf{\Omega}^{(y,y')} \|f\|_{\infty}
\end{aligned} \tag{127}$$

It follows that for the same x, f

$$L^{-2(k-j)} |(R_z f)(x)| \leq C M^{-1} L^{-2(k-j')} e^{-\frac{1}{2}\gamma_0 d} \mathbf{\Omega}^{(y,y')} \|f\|_{\infty} \tag{128}$$

Now consider $G_{k,\mathbf{\Omega},\omega}$ with $|\omega| = n$. By (102) and (128) we have for $x \in \Delta_y$ and $\text{supp} f \subset \Delta_{y'}$ with $y_0 = y, y_{n+1} = y'$

$$\begin{aligned}
|(G_{k,\mathbf{\Omega},\omega} f)(x)| &= \left| \left(h_{\omega_0} G_{k,\mathbf{\Omega}}(\tilde{\square}_{\omega_0}) h_{\omega_0} \right) R_{\omega_1} \cdots R_{\omega_n} f \right)(x) \Big| \\
&\leq \sum_{y_1, \dots, y_n} \left| \left(h_{\omega_0} G_{k,\mathbf{\Omega}}(\tilde{\square}_{\omega_0}) h_{\omega_0} \right) 1_{\Delta_{y_1}} R_{\omega_1} 1_{\Delta_{y_2}} \cdots 1_{\Delta_{y_n}} R_{\omega_n} f \right)(x) \Big| \\
&\leq C \left(C M^{-1} \right)^n L^{-2(k-j')} \sum_{y_1, \dots, y_n} \prod_{j=0}^n e^{-\frac{1}{2}\gamma_0 d} \mathbf{\Omega}^{(y_j, y_{j+1})} \|f\|_{\infty} \\
&\leq C \left(C M^{-1} \right)^n L^{-2(k-j')} e^{-\frac{1}{4}\gamma_0 d} \mathbf{\Omega}^{(y_j, y_{j+1})} \|f\|_{\infty}
\end{aligned} \tag{129}$$

In the last step we use

$$\sum_{j=0}^n d_{\mathbf{\Omega}}(y_j, y_{j+1}) \geq d_{\mathbf{\Omega}}(y, y') \quad (130)$$

to extract a factor $e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}}(y, y')}$, and then use (98) with $\delta = \frac{1}{2}\gamma_0$ repeatedly.

For convergence of the random walk expansion we have

$$|(G_{k, \mathbf{\Omega}} f)(x)| \leq \sum_{\omega} |(G_{k, \mathbf{\Omega}, \omega} f)(x)| \leq CL^{-2(k-j')} e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}}(y_j, y_{j+1})} \|f\|_{\infty} \sum_{n=0}^{\infty} \sum_{\omega: |\omega|=n} (CM^{-1})^n \quad (131)$$

But for each $\tilde{\square}_z$ there are at most $\mathcal{O}(L^2)$ cubes $\tilde{\square}_{z'}$ such that $\tilde{\square}_z \cap \tilde{\square}_{z'} \neq \emptyset$, and so there are at most $\mathcal{O}(L^{2n})$ paths with $|\omega| = n$. Thus for M sufficiently large the sum is bounded by $\sum_{n=0}^{\infty} (CL^2 M^{-1})^n \leq 2$. This establishes the bound on $G_{k, \mathbf{\Omega}}$.

For the bound on $\partial G_{k, \mathbf{\Omega}}$ there are terms with $\partial G_{k, \mathbf{\Omega}}(\tilde{\square}_{\omega_0})$ and we need the extra factor $L^{-(k-j)}$ to estimate it by (102). A similar remark applies to the Holder derivatives. This completes the proof.

As an application we give an estimate on $\phi_{k, \mathbf{\Omega}} = \phi_{k, \mathbf{\Omega}}(\phi, \Phi_{k, \mathbf{\Omega}})$ as defined in (42). With $\delta\Omega_j = \Omega_j - \Omega_{j+1}$ for $j = 1, \dots, k-1$ and $\delta\Omega_k = \Omega_k$ define

$$\|\Phi_{k, \mathbf{\Omega}}\|_{\infty} = \sup_{1 \leq j \leq k} \|\Phi_{j, \delta\Omega_j}\|_{\infty} \quad (132)$$

Lemma 2.7. *There is a constant C depending only on L such that on $\delta\Omega_j$*

$$|\phi_{k, \mathbf{\Omega}}|, \quad L^{-(k-j)} |\partial \phi_{k, \mathbf{\Omega}}|, \quad L^{-(1+\alpha)(k-j)} |\delta_{\alpha} \partial \phi_{k, \mathbf{\Omega}}| \leq C \left(\|\phi\|_{\infty} + \|\Phi_{k, \mathbf{\Omega}}\|_{\infty} \right) \quad (133)$$

Proof. $\phi_{k, \mathbf{\Omega}}$ is a sum of two terms. The first is

$$\begin{aligned} G_{k, \mathbf{\Omega}} Q_{k, \mathbf{\Omega}}^T \mathbf{a}^{(k)} \Phi_{k, \mathbf{\Omega}} &= \sum_{j'=1}^k a_{j'}^{(k)} G_{k, \mathbf{\Omega}} Q_{j'}^T \Phi_{j', \delta\Omega_{j'}} \\ &= \sum_{j'=1}^k \sum_{y' \in \delta\Omega_{j'}^{(j')}} a_{j'} L^{2(k-j')} G_{k, \mathbf{\Omega}} 1_{\Delta_{y'}} Q_{j'}^T \Phi_{j', \delta\Omega_{j'}} \end{aligned} \quad (134)$$

Then on $\Delta_y \subset \delta\Omega_j$, by (115) and (98) (note the cancellation of the factors $L^{2(k-j')}$ by (115))

$$\begin{aligned} \left| G_{k, \mathbf{\Omega}} Q_{k, \mathbf{\Omega}}^T \mathbf{a}^{(k)} \Phi_{k, \mathbf{\Omega}} \right| &\leq C \sum_{j'=1}^k \sum_{y' \in \delta\Omega_{j'}^{(j')}} e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}}(y, y')} \|Q_{j'}^T \Phi_{j', \delta\Omega_{j'}}\|_{\infty} \\ &= C \sum_{y'} e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}}(y, y')} \|\Phi_{k, \mathbf{\Omega}}\|_{\infty} \leq C \|\Phi_{k, \mathbf{\Omega}}\|_{\infty} \end{aligned} \quad (135)$$

The second term is for ϕ on Ω_1^c :

$$G_{k, \mathbf{\Omega}} [\Delta]_{\Omega_1, \Omega_1^c} \phi = \sum_{y' \in \delta\Omega_1^{(1)}} G_{k, \mathbf{\Omega}} 1_{\Delta_{y'}} [\Delta]_{\Omega_1, \Omega_1^c} \phi \quad (136)$$

Then on $\Delta_y \subset \delta\Omega_j$, by (115) and (98))

$$\left| G_{k, \mathbf{\Omega}} [\Delta]_{\Omega_1, \Omega_1^c} \phi \right| \leq C \sum_{y'} e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}}(y, y')} L^{-2(k-1)} \|[\Delta]_{\Omega_1, \Omega_1^c} \phi\|_{\infty} \leq C \|\phi\|_{\infty} \quad (137)$$

In the last step we used that for $\phi : \mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k} \rightarrow \mathbb{R}$ we have $\|\Delta\phi\|_\infty \leq \mathcal{O}(1)L^{2k}\|\phi\|_\infty$. Combining the two bounds gives the bound on $\phi_{k,\mathbf{\Omega}}$. The bounds on the derivatives are similar.

Variations:

(A.) A local version of (133) will also be useful. This says for $L^{-(k-j)}$ cubes Δ_y in $\delta\Omega_j$

$$\begin{aligned} & \left| 1_{\Delta_y} \phi_{k,\mathbf{\Omega}} \left(1_{\Delta_{y'}} (\phi, \Phi_{k,\mathbf{\Omega}}) \right) \right|, \quad L^{-(k-j)} \left| 1_{\Delta_y} \partial \phi_{k,\mathbf{\Omega}} \left(1_{\Delta_{y'}} (\phi, \Phi_{k,\mathbf{\Omega}}) \right) \right|, \\ & L^{-(1+\alpha)(k-j)} \left| 1_{\Delta_y} \delta_\alpha \partial \phi_{k,\mathbf{\Omega}} \left(1_{\Delta_{y'}} (\phi, \Phi_{k,\mathbf{\Omega}}) \right) \right| \leq C e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}}(y,y')} \left(\|\phi\|_\infty + \|\Phi_{k,\mathbf{\Omega}}\|_\infty \right) \end{aligned} \quad (138)$$

The follows since only one term in the final sum over y' contributes.

(B.) We can introduce a weakening parameter $0 \leq s_\square \leq 1$ for each $L^{-(k-j)}M$ square \square in $\delta\Omega_j$ and all $1 \leq j \leq k$. Define

$$s_\omega = \prod_{\square \subset X_\omega} s_\square \quad X_\omega = \bigcup_{j=1}^n \tilde{\square}_{\omega_j} \quad (139)$$

and define $G_{k,\mathbf{\Omega}}(s)$ by

$$G_{k,\mathbf{\Omega}}(s) = \sum_{\omega} s_\omega G_{k,\mathbf{\Omega},\omega} \quad (140)$$

In the basic convergence proof of the lemma we do not need all of the M^{-1} , a factor $M^{-1/2}$ would do. Thus if s_\square is complex and $|s_\square| \leq e^{\kappa_1}$ we have an extra factor $\prod_\square |s_\square| M^{-1/2}$ to estimate. This is less than one provided $e^{\kappa_1} \leq M^{1/2}$ which we assume.

All the above results hold with $|s_\square| \leq e^{\kappa_1}$. In particular theorem 2.2 holds with $G_{k,\mathbf{\Omega}}$ replaced by $G_{k,\mathbf{\Omega}}(s)$. We also change $\phi_{k,\mathbf{\Omega}}$ to $\phi_{k,\mathbf{\Omega}}(s)$ by replacing $G_{k,\mathbf{\Omega}}$ by $G_{k,\mathbf{\Omega}}(s)$. Then $\phi_{k,\mathbf{\Omega}}(s)$ satisfies the bounds of lemma 2.7 and (138)

(C.) Similar results hold for the Green's function $G_{k+1,\mathbf{\Omega}^+}^0$ defined in (56). This has a random walk expansion which is a scaling up of the expansion (114) for $k+1$. In this case the statement of the theorem says that for $\Delta_y \subset \delta\Omega_j$ and $\Delta_{y'} \subset \delta\Omega_{j'}$:

$$\begin{aligned} & \left| 1_{\Delta_y} G_{k+1,\mathbf{\Omega}^+}^0 1_{\Delta_{y'}} f \right|, \quad L^{-(k-j)} \left| 1_{\Delta_y} \partial G_{k+1,\mathbf{\Omega}^+}^0 1_{\Delta_{y'}} f \right|, \quad L^{-(1+\alpha)(k-j)} \left| 1_{\Delta_y} \delta_\alpha \partial G_{k+1,\mathbf{\Omega}^+}^0 1_{\Delta_{y'}} f \right| \\ & \leq C L^{-2(k-j')} e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}^+}(y,y')} \|f\|_\infty \end{aligned} \quad (141)$$

Here $j = 1, \dots, k+1$ with $\delta\Omega_{k+1} = \Omega_{k+1}$. Now Δ_y in Ω_{k+1} is an L -cube. Also $d_{\mathbf{\Omega}^+}(y, y')$ is defined as in (97) but with the sum up to $k+1$, so in Ω_{k+1} paths are weighted by L^{-2} . As in (133) the associated fields $\phi_{k+1,\mathbf{\Omega}^+}^0$ defined in (55) satisfy on $\delta\Omega_j$

$$|\phi_{k+1,\mathbf{\Omega}^+}^0|, \quad L^{-(k-j)} |\partial \phi_{k+1,\mathbf{\Omega}^+}^0|, \quad L^{-(1+\alpha)(k-j)} |\delta_\alpha \partial \phi_{k+1,\mathbf{\Omega}^+}^0| \leq C \left(\|\phi\|_\infty + \|\Phi_{k+1,\mathbf{\Omega}^+}\|_\infty \right) \quad (142)$$

This can be understood as (133) for $k+1$ scaled up by L .

Finally one can introduce weakening parameters $\{s_\square\}$, replacing $G_{k+1,\mathbf{\Omega}^+}^0$ with $G_{k+1,\mathbf{\Omega}^+}^0(s)$ and $\phi_{k+1,\mathbf{\Omega}^+}^0$ with $\phi_{k+1,\mathbf{\Omega}^+}^0(s)$, and obtain bounds of the same form.

3 The full expansion

3.1 definitions and notation

In this section and the next we introduce the concepts we need to state the main theorem. A basic parameter is the scaled coupling constant

$$\lambda_k = \lambda_k^N = L^{-(N-k)} \lambda \quad (143)$$

which satisfies $\lambda_k = L^k \lambda_0$. This is our effective coupling constant after k renormalization group steps. We always assume λ_k is sufficiently small depending on the parameter L, M , and in particular $\log(-\lambda_k) \geq 1$.

3.1.1 small field regions

At the k^{th} stage of the iteration we will introduce not one but two new small field regions Ω_k, Λ_k and the pair is denoted $\Pi_k = (\Omega_k, \Lambda_k)$. Each will be associated with the introduction of characteristic functions in a manner yet to be explained. After k steps there is a sequence of regions with $\Omega_0 = \emptyset$ and

$$\mathbf{\Pi} = (\Pi_0, \Pi_1, \dots, \Pi_n) = (\Lambda_0, \Omega_1, \Lambda_1, \Omega_2, \Lambda_2, \dots, \Omega_k, \Lambda_k) \quad (144)$$

These are decreasing:

$$\Lambda_0 \supset \Omega_1 \supset \Lambda_1 \supset \Omega_2 \supset \Lambda_2 \supset \dots \supset \Omega_k \supset \Lambda_k \quad (145)$$

All these regions are subsets of \mathbb{T}_{M+N-k}^{-k} and Ω_j, Λ_j are unions of $L^{-(k-j)}M$ cubes. We also use the notation

$$\mathbf{\Omega} = (\Omega_1, \Omega_2, \dots, \Omega_k) \quad \mathbf{\Lambda} = (\Lambda_0, \Lambda_1, \Lambda_2, \dots, \Lambda_k) \quad (146)$$

We require much stronger separation conditions than those defined in the previous section. Define

$$r_k = r(\lambda_k) = (-\log \lambda_k)^r = ((N-k) \log L - \log \lambda)^r \quad (147)$$

for some positive integer r . We assume always λ_k is small so $-\log \lambda_k > 0$ is large and r_k is large and decreasing in k . The separation requirement is that

$$d((\bar{\Lambda}_{j-1})^c, \Omega_j) \geq 5[r_j]L^{-(k-j)}M \quad d(\Omega_j^c, \Lambda_j) \geq 5[r_j]L^{-(k-j)}M \quad (148)$$

where here $\bar{\Lambda}_{j-1}$ is all $L^{-(k-j)}M$ cubes intersecting Λ_{j-1} .

There are some special cases. It may be that some region is the full torus \mathbb{T}_{M+N-k}^{-k} . In this case all larger regions are also the full torus and for these there is no separation requirement. It may also be that some region is empty. In this case all smaller regions are also empty and for these there is no separation requirement.

3.1.2 polymers

Recall that a polymer $X \in \mathcal{D}_k$ is a connected union of M -cubes in \mathbb{T}_{M+N-k}^{-k} .

A variation is a polymer with holes. Given a final small field region Ω_k (not necessarily connected) on the same torus, suppose the large field region Ω_k^c has connected components $\Omega_{k,\alpha}^c$ (the holes). We define a subset of \mathcal{D}_k by

$$\mathcal{D}_k(\text{mod } \Omega_k^c) = \{X \subset \mathcal{D}_k : \text{for all } \alpha \text{ either } \Omega_{k,\alpha}^c \subset X \text{ or } \Omega_{k,\alpha}^c, X \text{ are disjoint}\} \quad (149)$$

We associate with any $X \in \mathcal{D}_k(\text{mod } \Omega_k^c)$ a linear distance $d_M(X, \text{mod } \Omega_k^c)$ on scale M defined by

$$M d_M(X, \text{mod } \Omega_k^c) = \inf_{\tau \text{ on } X} \ell(\tau) \quad (150)$$

where the infimum is over all continuum tree graphs τ contained in X and intersecting every M -cube in $X \cap \Omega_k$, and $\ell(\tau)$ is the length of τ . Thus $d_M(X, \text{mod } \Omega_k^c)$ measures the size of the components of $X \cap \Omega_k$ and the distances between these components. But $d_M(X, \text{mod } \Omega_k^c)$ does not measure the bulk of $X \cap \Omega_k^c$. The idea is that decay in these regions (holes) will be taken care of elsewhere. If $X \subset \Omega_k$ then $d_M(X, \text{mod } \Omega_k^c) = d_M(X)$ where $d_M(X)$ is the infimum of the lengths of continuum tree graphs intersecting *every* M -cube in X . In general $d_M(X, \text{mod } \Omega_k^c) \leq d_M(X)$. For any M -cube $\square \in \Omega_k$ we have for a universal constants κ_0, K_0

$$\sum_{X \in \mathcal{D}_k(\text{mod } \Omega_k^c), X \supset \square} e^{-\kappa_0 d_M(X, \text{mod } \Omega_k^c)} \leq K_0 \quad (151)$$

See appendix E for the proof.

Another variation is multiscale polymers $\mathcal{D}_{k,\mathbf{\Omega}}$. An element X of $\mathcal{D}_{k,\mathbf{\Omega}}$ is a connected subset of \mathbb{T}_{M+N-k}^{-k} with that $X \cap \delta\Omega_j$ is a union of $L^{-(k-j)}$ cubes. Let

$$|X|_{\mathbf{\Omega}} = \sum_{j=1}^k |X \cap \delta\Omega_j|_{L^{-(k-j)} M} \quad (152)$$

be the total number of blocks in X . Then for any elementary cube $\square \subset \mathcal{D}_{k,\mathbf{\Omega}}$ and κ_* large enough

$$\sum_{X \in \mathcal{D}_{k,\mathbf{\Omega}}, X \supset \square} e^{-\kappa_* |X|_{\mathbf{\Omega}}} \leq e^{-\frac{1}{2}\kappa_*} \quad (153)$$

See appendix D for the proof.

We will also consider \mathcal{D}_{k+1}^0 which is connected unions of LM cubes. Also for $\mathbf{\Omega}^+ = (\Omega_1, \dots, \Omega_{k+1})$ with Ω_{k+1} a union of LM cubes, we define $\mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1})$ and $\mathcal{D}_{k+1,\mathbf{\Omega}^+}^0$ as above. These are still in \mathbb{T}_{M+N-k}^{-k} but they will scale to $\mathcal{D}_{k+1}, \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1})$ and $\mathcal{D}_{k+1,\mathbf{\Omega}^+}$ on $\mathbb{T}_{M+N-k-1}^{-k-1}$.

3.1.3 localized functionals

As discussed in section 2.1, associated with the regions $(\Omega_1, \Omega_2, \dots, \Omega_k)$ are fundamental fields

$$\Phi_{k,\mathbf{\Omega}} = (\Phi_{1,\delta\Omega_1}, \dots, \Phi_{k-1,\delta\Omega_{k-1}}, \Phi_{k,\Omega_k}) \quad (154)$$

where $\delta\Omega_j = \Omega_j - \Omega_{j+1}$ and $\Phi_{j,\delta\Omega_j}$ is defined on the subset $\delta\Omega_j^{(j)}$. We want to consider functions of the fields of the form $H(X, \Phi_{k,\mathbf{\Omega}})$ with $X \in \mathcal{D}_k(\text{mod } \Omega_k^c)$ and the property that they only depend on $\Phi_{k,\mathbf{\Omega}}$ in X . For bounded fields these will satisfy bounds like

$$|H(X, \Phi_{k,\mathbf{\Omega}})| \leq \text{const} \exp\left(-\kappa_0 d_M(X, \text{mod } \Omega_k^c)\right) \quad (155)$$

We also consider sums of these denoted for any union of M -cubes Λ by ³

$$H(\Lambda) = \sum_{X \subset \Lambda} H(X) \quad (156)$$

For boundary terms, denoted by $B(\Lambda)$ or some such, we employ a different convention summing over polymers X that cross Λ . We write

$$B(\Lambda) = \sum_{X \# \Lambda} B(X) \quad (157)$$

³This is not a very good notation, since if Λ is connected this could refer to the single function $H(\Lambda)$ rather than the sum. One should really denote the sum by a different symbol. But the notation is already overburdened, so we instead we adopt the convention that if the region is given by a Greek letter like Λ then we mean the sum

where $X \# \Lambda$ means X crosses Λ or

$$X \# \Lambda \iff X \cap \Lambda \neq \emptyset \text{ and } X \cap \Lambda^c \neq \emptyset \quad (158)$$

There are also the associated smeared fields $\phi_{k,\Omega}$ on \mathbb{T}_{M+N-k}^{-k} as defined in (42) and we will also want to consider functionals of the fields of the form $H(X, \phi_{k,\Omega})$ or $H(X, \Phi_{k,\Omega}, \phi_{k,\Omega})$. These are also required to depend on the indicated fields in X . In this case the functional depends on the fundamental fields Φ_{Ω} outside of X , but only very weakly. In fact it will be useful to have a stricter localization. This will be accomplished by introducing modifications of $\phi_{k,\Omega}$ with stricter localization, which we now explain in several steps.

3.1.4 averaging operators again

First revisit the averaging operator $Q_{k,\Omega}$, also denoted $Q_{\Omega, \mathbb{T}^{-k}}$. For ϕ on \mathbb{T}_{M+N-k}^{-k} we defined in (27)

$$Q_{k,\Omega}\phi = Q_{\Omega, \mathbb{T}^{-k}}\phi = ([Q_1\phi]_{\delta\Omega_1}, \dots, [Q_{k-1}\phi]_{\delta\Omega_{k-1}}, [Q_k\phi]_{\Omega_k}) \quad (159)$$

As a completion for this averaging we define on $Q_{k,\Omega}\phi$, or more generally any multiscale field $\Phi_{k,\Omega}$ of the form (154) the averaging operator

$$Q_{\mathbb{T}^0, \Omega}\Phi_{k,\Omega} = (Q_{k-1}\Phi_{1, \delta\Omega_1}, \dots, Q_1\Phi_{k-1, \delta\Omega_{k-1}}, \Phi_{k, \Omega_k}) \quad (160)$$

Then we have

$$Q_{\mathbb{T}^0, \Omega}Q_{\Omega, \mathbb{T}^{-k}}\phi = Q_{\mathbb{T}^0, \mathbb{T}^{-k}}\phi = Q_k\phi \quad (161)$$

3.1.5 buffers

Let X be a union of M cubes in \mathbb{T}_{M+N-k}^{-k} . We define a minimal buffer

$$\Omega(X) = (\Omega_1(X), \Omega_2(X), \dots, \Omega_k(X)) \quad (162)$$

around X as follows. The region $\Omega_k(X)$ is X with $R = \mathcal{O}(1)$ layers of M -cubes added. Then for $j = k-1, \dots, 1$ we successively define Ω_j to be Ω_{j+1} with R layers of $L^{-(k-j)}M$ cubes added. Then $X \subset \Omega_k(X) \subset \Omega_{k-1}(X) \subset \dots \subset \Omega_1(X)$ and the separation condition (96) is satisfied. The regions $\Omega(X)$ are not generated in the same way as the Ω in the main expansion, and if X is small a more typical picture of $\Omega(X)$ is shown in figure 2.

We also define

$$X^{\sim n} = X \text{ enlarged by } n \text{ layers of } M \text{ blocks} \quad (163)$$

and $\tilde{X} = X^{\sim}$ is the case $n = 1$. Then we have

$$\Omega_1(X) \subset X^{\sim 2R} \quad (164)$$

We also consider larger buffers defining

$$X^* = X \text{ enlarged by } [r_k] \text{ layers of } M \text{ blocks} = X^{\sim [r_k]} \quad (165)$$

and $X^{2*} = X^{**}$, $X^{3*} = X^{***}$, etc. We also define

$$X^{\natural} = X_k \text{ shrunk by } [r_k] \text{ layers of } M \text{ blocks} = ((X^c)^*)^c \quad (166)$$

and $X^{2\natural} = X^{\natural\natural}$, $X^{3\natural} = X^{\natural\natural\natural}$, etc.

For a single cube $\square^{\natural*} = \square = \square^{*\natural}$, but in general

$$X^{\natural*} \subset X \subset X^{*\natural} \quad (167)$$

The definitions of X^{\sim} , X^* , X^{\natural} vary with scale. If X is specified as a union of $L^{-(k-j)}M$ blocks, then the enlargements also are taken with $L^{-(k-j)}M$ blocks.

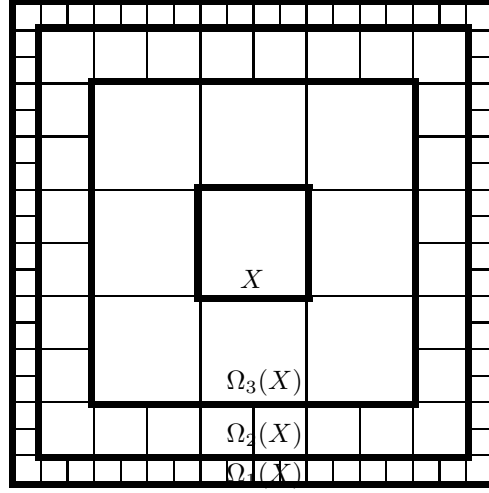


Figure 2: A minimal buffer $X \subset \Omega_3(X) \subset \Omega_2(X) \subset \Omega_1(X)$

3.1.6 sharply localized fields

1. After k steps the current small field region will be Λ_k and instead of $\phi_{k,\Omega}$ we will want to consider a field more sharply localized in this region; but still not strictly localized which would be too singular. These will be based on the buffer $\Omega(\Lambda_k^*)$ which provides a gradual transition. We would like to consider the field $\phi_{k,\Omega(\Lambda^*)}$, but it has to be expressed in terms of the fundamental variables $\Phi_{k,\Omega}$ and these are not compatible with the buffer. The buffer $\Omega(\Lambda^*)$ is smaller since $\Omega_1(\Lambda_k^*) \subset \Omega_k$. Figure 3 might help in remembering the ordering of these regions.

To remedy this consider the averaging operators $Q_{\mathbb{T}^0, \Omega(\Lambda_k^*)}$ which take functions $\Phi_{\Omega(\Lambda_k^*)}$ to functions on the unit lattice. Then the adjoint $Q_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T$ takes functions on the unit lattice to functions $\Phi_{\Omega(\Lambda_k^*)}$. It has the form

$$Q_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k = \left([Q_{k-1}^T \Phi_k]_{\delta\Omega_1(\Lambda_k^*)}, \dots, [Q_1^T \Phi_k]_{\delta\Omega_{k-1}(\Lambda_k^*)}, \Phi_{k, \Omega_k(\Lambda_k^*)} \right) \quad (168)$$

where $[Q_{k-j}^T \Phi_k]_{\delta\Omega_j(\Lambda_k^*)}$ is defined on $\delta\Omega_j^{(j)} \subset \mathbb{T}_{M+N-k}^{-(k-j)}$. This is equal to Φ_k everywhere, but on an increasingly fine lattice as one moves away from Λ_k^* . In $\Omega_1(\Lambda_k^*)^c \subset \mathbb{T}_{M+N-k}^{-k}$ we can take $Q_k^T \Phi_k$. The combination is denoted

$$\tilde{Q}_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k = \left([Q_k^T \Phi_k]_{\Omega_1(\Lambda_k^*)^c}, Q_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k \right) \quad (169)$$

and we may consider

$$\phi_{k, \Omega(\Lambda_k^*)} \equiv \phi_{k, \Omega(\Lambda_k^*)} \left(\tilde{Q}_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k \right) \quad (170)$$

Note that this field $\phi_{k, \Omega(\Lambda_k^*)}$ is defined on a neighborhood of $\Omega_1(\Lambda_k^*) \subset \Omega_k$. and depends only on Φ_k .

2. Another case is a field defined and localized by $\Lambda_{k-1}, \Omega_k, \Lambda_k$ as above. Suppose $\Omega(\Lambda_{k-1}^*)$ is a buffering sequence of length $k-1$, defined as above, but finishing with unions of $L^{-1}M$ cubes. We adjoin $\Omega_k \subset \Lambda_{k-1}$ and define the sequence $(\Omega(\Lambda_{k-1}^*), \Omega_k)$ of length k finishing with unions

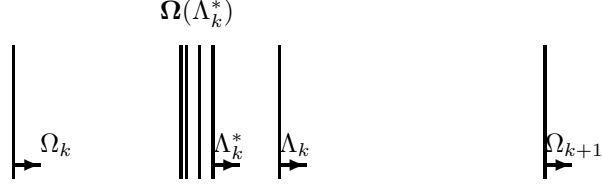


Figure 3: ordering of regions-I

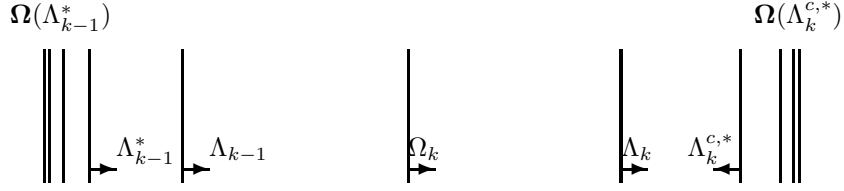


Figure 4: ordering of regions-II

of M cubes. Another sequence of length k is $\Omega(\Lambda_k^{c,*})$. We combine them and form ⁴

$$\Omega(\Lambda_{k-1}, \Omega_k, \Lambda_k) \equiv \left(\Omega(\Lambda_{k-1}^*), \Omega_k \right) \cap \Omega(\Lambda_k^{c,*}) \quad (171)$$

Now define a field $\phi_{k, \Omega(\Lambda_{k-1}, \Omega_k, \Lambda_k)}$ roughly localized in $\Lambda_{k-1}^* \cap \Lambda_k^{c,*}$ as

$$\phi_{k, \Omega(\Lambda_{k-1}, \Omega_k, \Lambda_k)} \left(\left[\tilde{Q}_{\mathbb{T}^{-1}}^T, \Omega(\Lambda_{k-1}^*) \Phi_{k-1} \right]_{\Omega_k^c}, \left[\tilde{Q}_{k, \mathbb{T}^0}^T, \Omega(\Lambda_k^{c,*}) \Phi_k \right]_{\Omega_k} \right) \quad (172)$$

Note that in $\Lambda_{k-1}^* \cap \Lambda_k^{c,*}$ the field in parentheses is just $(\Phi_{k-1, \delta\Omega_{k-1}}, \Phi_{k, \Omega_k})$. The ordering of the regions is illustrated in figure 4.

3. We also need still more localized fields. After k steps, let \square be an M -cube in Ω_k . We consider the buffer $\Omega(\square)$ and want to define $\phi_{k, \Omega(\square)}$. If \square is well inside Ω_k in the sense that $\square^{\sim(2R+1)} \subset \Omega_k$, then the buffer $\Omega(\square)$ is also in Ω_k and we can define $\phi_{k, \Omega(\square)}$ as a function of Φ_k alone by

$$\phi_{k, \Omega(\square)} = \phi_{k, \Omega(\square)} \left(\tilde{Q}_{\mathbb{T}^0, \Omega(\square)}^T \Phi_k \right) \quad (173)$$

If however \square is near the boundary of Ω_k then $\phi_{k, \Omega(\square)}$ depends on Φ_{k-1} as well. In this case we define

$$\phi_{k, \Omega(\square)} = \phi_{k, \Omega(\square)} \left(\tilde{Q}_{k, \mathbb{T}^0, \Omega(\square)}^T (Q \Phi_{k-1, \delta\Omega_{k-1}}, \Phi_{k, \Omega_k}) \right) \quad (174)$$

⁴ In general let $\Omega, \tilde{\Omega}$ be sequences of the form (11), and suppose that Ω_k^c and $\tilde{\Omega}_k^c$ are disjoint. Then we can define a new sequence

$$\Omega \cap \tilde{\Omega} = (\Omega_1 \cap \tilde{\Omega}_1, \dots, \Omega_k \cap \tilde{\Omega}_k)$$

If $\Omega, \tilde{\Omega}$ satisfy the separation condition (96) then so does $\Omega \cap \tilde{\Omega}$. The increments for $\Omega \cap \tilde{\Omega}$ are $\delta\Omega_j \cup \delta\tilde{\Omega}_j$.

In passing from k to $k+1$ we will introduced Φ_{k+1} in Ω_{k+1} and Φ_k will be eliminated on $\delta\Omega_k = \Omega_k - \Omega_{k+1}$. Then for $\square \in \delta\Omega_k$ we define instead of $\phi_{k,\Omega(\square)}$

$$\phi'_{k,\Omega(\square)} = \phi_{k,\Omega(\square)} \left(\tilde{Q}_{k,\mathbb{T}^0,\Omega(\square)}^T (Q\Phi_{k-1,\delta\Omega_{k-1}}, \Phi_{k,\delta\Omega_k}, Q^T\Phi_{k+1,\Omega_{k+1}}) \right) \quad (175)$$

This is the same as $\phi_{k,\Omega(\square)}$ away from the boundary of $\delta\Omega_k$

3.2 bounds on fields

As a further preliminary we define some small field regions. A basic parameter here is

$$p_k = p(\lambda_k) = (-\log \lambda_k)^p = ((N-k)\log L - \log \lambda)^p \quad (176)$$

for some positive integer p larger than r .

We start with some restrictions which will be forced by our characteristic functions.

Definition 3.1. For $\square \subset \Omega_k$, $\mathcal{S}_k(\square)$ is all $(\Phi_{k-1,\Omega_{k-1}}, \Phi_{k,\Omega_k})$ such that

$$\begin{aligned} |\Phi_k - Q_k \phi_{k,\Omega(\square)}| &\leq p_k & \text{on } \tilde{\square} \cap \Omega_k \\ |\partial \phi_{k,\Omega(\square)}| &\leq p_k & \text{on } \tilde{\square} \\ |\phi_{k,\Omega(\square)}| &\leq p_k \lambda_k^{-1/4} & \text{on } \tilde{\square} \end{aligned} \quad (177)$$

If X is a union of M -cubes

$$\mathcal{S}_k(X) = \bigcap_{\square \subset X} \mathcal{S}_k(\square) \quad (178)$$

We say that \square is *well-inside* Ω_k if $\square \subset^{(2R+1)} \Omega_k$. In this case $\mathcal{S}_k(\square)$ is a condition on Φ_k alone, and the bounds are more or less equivalent to bounds on $\Phi_k, \partial\Phi_k$ for we have the following result: ⁵

Lemma 3.1. Let \square be well-inside Ω_k , and suppose $\bar{\mu}_k \leq \mathcal{O}(1)\lambda_k^{\frac{1}{4}}$.

1. If $\Phi_k \in \mathcal{S}_k(\square)$ then on $\tilde{\square}$

$$|\Phi_k| \leq 2p_k \lambda_k^{-\frac{1}{4}} \quad |\partial\Phi_k| \leq 3p_k \quad (179)$$

2. Conversely if on $\square \sim^{(2R+1)}$

$$|\Phi_k| \leq p_k \lambda_k^{-\frac{1}{4}} \quad |\partial\Phi_k| \leq p_k \quad (180)$$

then $\Phi_k \in C\mathcal{S}_k(\square)$, i.e. the bounds (177) hold with a constant C on the right.

Proof. [2], [8]

(1.) For $x \in \tilde{\square}$

$$|\Phi_k(x)| \leq |\Phi_k(x) - Q_k \phi_{k,\Omega(\square)}(x)| + |Q_k \phi_{k,\Omega(\square)}(x)| \leq 2p_k \lambda_k^{-1/4} \quad (181)$$

Also for $x, x + e_\mu \in \tilde{\square}$

$$\begin{aligned} |(\partial_\mu \Phi_k)(x)| &= |(\Phi_k)(x + e_\mu) - (\Phi_k)(x)| \\ &\leq |Q_k \phi_{k,\Omega(\square)}(x + e_\mu) - Q_k \phi_{k,\Omega(\square)}(x)| + 2p_k \\ &\leq \sup_{x \in \tilde{\square}} |\partial \phi_{k,\Omega(\square)}(x)| + 2p_k \leq 3p_k \end{aligned} \quad (182)$$

⁵ It might seem more straightforward to work directly with bounds on $\Phi_k, \partial\Phi_k$. The conditions (177) turn out to be more convenient since they correspond directly to pieces of the action, and behave better under iteration.

This proves the first part.

(2.) We need $|\phi_{k,\mathbf{\Omega}(\square)}| \leq C\lambda_k^{-1/4}p_k$ on $\tilde{\square}$. Since our assumptions imply $(\square')^{\sim 2} \subset \Omega_k(\square)$ for all $\square' \subset \tilde{\square}$ we can use (133) to estimate on $\tilde{\square}$ that $|\phi_{k,\mathbf{\Omega}(\square)}| \leq C\|\Phi_k\|_\infty$ with the supremum over $\square^{\sim(2R+1)}$. Then the result follows from the bound on Φ_k . For the derivative we can get the same bound, but we need the better bound $|\partial\phi_{k,\mathbf{\Omega}(\square)}| \leq Cp_k$ on $\tilde{\square}$.

For this we use the following identity. Let y be unit lattice point in $\tilde{\square}$ and take x in a neighborhood of Δ_y . Then the claim is that

$$\begin{aligned} \left[\phi_{k,\mathbf{\Omega}(\square)} \left(\tilde{Q}_{\mathbb{T}^0,\mathbf{\Omega}(\square)}^T \Phi_k \right) \right] (x) &= \left[\phi_{k,\mathbf{\Omega}(\square)} \left(\tilde{Q}_{\mathbb{T}^0,\mathbf{\Omega}(\square)}^T (\Phi_k - \Phi_k(y)) \right) \right] (x) \\ &\quad + \left[1 - \bar{\mu}_k G_{k,\mathbf{\Omega}(\square)} \cdot 1 \right] (x) \Phi_k(y) \end{aligned} \quad (183)$$

Indeed the identity holds if the second term on the right is $[\phi_{k,\mathbf{\Omega}(\square)}(\tilde{Q}_{\mathbb{T}^0,\mathbf{\Omega}(\square)}^T 1)](x)\Phi_k(y)$ which is the same as $[\phi_{k,\mathbf{\Omega}(\square)}(1)](x)\Phi_k(y)$. However since $[-\Delta]_\Omega \cdot 1 = \Delta_{\Omega,\Omega^c} \cdot 1$ we have

$$\left[-\Delta + \bar{\mu}_k + Q_{k,\mathbf{\Omega}(\square)}^T \mathbf{a} Q_{k,\mathbf{\Omega}(\square)} \right]_{\Omega_1(\square)} \cdot 1 = \left(\Delta_{\Omega_1(\square),\Omega_1^c(\square)} + \bar{\mu}_k + Q_{k,\mathbf{\Omega}(\square)}^T \mathbf{a} \right) \cdot 1 \quad (184)$$

and so

$$G_{k,\mathbf{\Omega}(\square)} \left(\bar{\mu}_k + Q_{k,\mathbf{\Omega}(\square)}^T \mathbf{a} + \Delta_{\Omega_1(\square),\Omega_1^c(\square)} \right) \cdot 1 = 1 \quad (185)$$

Therefore

$$\phi_{k,\mathbf{\Omega}(\square)}(1) = G_{k,\mathbf{\Omega}(\square)} \left((Q_{k,\mathbf{\Omega}(\square)}^T \mathbf{a} + \Delta_{\Omega_1(\square),\Omega_1^c(\square)}) \cdot 1 \right) = 1 - \bar{\mu}_k G_{k,\mathbf{\Omega}(\square)} \cdot 1 \quad (186)$$

which gives the result (183).

On $\tilde{\square} \subset \Omega_k(\square)$ we have $|\partial G_{k,\mathbf{\Omega}(\square)} \cdot 1| \leq C$ by (115). Thus the derivative of the second term in (183) is bounded by

$$\left| \partial(1 - \bar{\mu}_k G_{k,\mathbf{\Omega}(\square)} \cdot 1) \Phi_k(y) \right| \leq C \bar{\mu}_k p_k \lambda_k^{-\frac{1}{4}} \leq Cp_k \quad (187)$$

By (133) for derivatives and the bound on $\partial\Phi_k$, the derivative of the first term in (183) is bounded on $\tilde{\square} \subset \Omega_k(\square)$ by

$$\begin{aligned} &\sum_{y'} \left| \left[\partial \phi_{k,\mathbf{\Omega}(\square)} \left(1_{\Delta_{y'}} \tilde{Q}_{\mathbb{T}^0,\mathbf{\Omega}(\square)}^T (\Phi_k - \Phi_k(y)) \right) \right] (x) \right| \\ &\leq C \sum_{y'} e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}(\square)}(y,y')} \| 1_{\Delta_{y'}} \tilde{Q}_{\mathbb{T}^0,\mathbf{\Omega}(\square)}^T (\Phi_k - \Phi_k(y)) \|_\infty \\ &\leq C \sum_{y'} e^{-\frac{1}{4}\gamma_0 d_{\mathbf{\Omega}(\square)}(y,y')} (d(y,y') + 1) p_k \leq Cp_k \end{aligned} \quad (188)$$

The last holds since $d(y,y') \leq d_{\mathbf{\Omega}(\square)}(y,y')$. Thus we end with $|\partial\phi_{k,\mathbf{\Omega}(\square)}(x)| \leq Cp_k$.

Finally we need $|\Phi_k - Q_k \phi_{k,\mathbf{\Omega}(\square)}| \leq Cp_k$ on $\tilde{\square}$. Again let y be a unit lattice point in $\tilde{\square}$ and consider $\Phi_k(y) - (Q_k \phi_{k,\mathbf{\Omega}(\square)})(y)$. Insert the expression (183) for $\phi_{k,\mathbf{\Omega}(\square)}$. Terms arising from the first term in (183) are bounded by Cp_k as in (188) (now for $\phi_{k,\mathbf{\Omega}(\square)}$ not $\partial\phi_{k,\mathbf{\Omega}(\square)}$). The remaining terms are

$$\Phi_k(y) - \left[Q(1 - \bar{\mu}_k G_{k,\mathbf{\Omega}(\square)} \cdot 1) \right] (y) \Phi_k(y) = \bar{\mu}_k (Q G_{k,\mathbf{\Omega}(\square)} \cdot 1)(y) \Phi_k(y) \quad (189)$$

Since $|G_{k,\mathbf{\Omega}(\square)} \cdot 1| \leq C$, this is bounded by Cp_k as in (187). This completes the proof.

Here is a variation in which \square is allowed to approach the boundary of Ω_k .

Lemma 3.2. Let $\square \subset \Omega_k$ and suppose $\bar{\mu}_k \leq \mathcal{O}(1)\lambda_k^{\frac{1}{4}}$.

1. If $(\Phi_{k-1, \Omega_{k-1}}, \Phi_{k, \Omega_k}) \in \mathcal{S}_k(\square)$ then on $\tilde{\square} \cap \Omega_k$

$$|\Phi_k| \leq 2p_k \lambda_k^{-\frac{1}{4}} \quad |\partial \Phi_k| \leq 3p_k \quad (190)$$

2. Conversely if on $\square \sim (2R+1)$ the field $\Phi_k^\# \equiv (Q\Phi_{k-1, \Omega_{k-1}}, \Phi_{k, \Omega_k})$ satisfies

$$|\Phi_k^\#| \leq p_k \lambda_k^{-\frac{1}{4}} \quad |\partial \Phi_k^\#| \leq p_k \quad (191)$$

then $(\Phi_{k-1, \Omega_{k-1}}, \Phi_{k, \Omega_k}) \in C\mathcal{S}_k(\square)$.

This is proved as in the previous lemma, but now $\phi_{k, \Omega(\square)}$ is $\phi_{k, \Omega(\square)}(\tilde{Q}_{k, \mathbb{T}^0, \Omega(\square)}^T \Phi_k^\#)$ rather than $\phi_{k, \Omega(\square)}(\tilde{Q}_{k, \mathbb{T}^0, \Omega(\square)}^T \Phi_k)$.

Next we introduce an analyticity domain for our fundamental fields.

Definition 3.2. Let δ be a fixed small positive number. Let \square be an M -cube in $\Omega_k \subset \mathbb{T}_{M+N-k}^{-k}$. Define $\mathcal{P}_k(\square, \delta)$ to be all complex-valued $(\Phi_{k-1, \delta\Omega_{k-1}}, \Phi_{k, \Omega_k})$ satisfying:

$$\begin{aligned} |\Phi_k - Q_k \phi_{k, \Omega(\square)}| &\leq \lambda_k^{-\frac{1}{4}-\delta} \quad \text{on } \tilde{\square} \cap \Omega_k \\ |\partial \phi_{k, \Omega(\square)}| &\leq \lambda_k^{-\frac{1}{4}-\delta} \quad \text{on } \tilde{\square} \\ |\phi_{k, \Omega(\square)}| &\leq \lambda_k^{-\frac{1}{4}-\delta} \quad \text{on } \tilde{\square} \end{aligned} \quad (192)$$

For $X \subset \Omega_k$ define

$$\mathcal{P}_k(X, \delta) = \bigcap_{\square \subset X} \mathcal{P}_k(\square, \delta) \quad (193)$$

If \square is well inside $\delta\Omega_k$ then the bounds (192) are a condition on Φ_{k, Ω_k} alone. As in lemma 3.1 if the fields are in $\mathcal{P}_k(\square)$ then on $\tilde{\square} \cap \Omega_k$

$$|\Phi_k| \leq 2p_k \lambda_k^{-\frac{1}{4}-\delta} \quad |\partial \Phi_k| \leq 3\lambda_k^{-\frac{1}{4}-\delta} \quad (194)$$

Note also that $\mathcal{S}_k(\square) \subset \mathcal{P}_k(\square, \delta)$.

Once we have introduced Ω_{k+1}, Φ_{k+1} we use a modified definition:

Definition 3.3. Let $\mathcal{P}'_k(\square, \delta)$ to be all complex-valued $(\Phi_{k-1, \delta\Omega_{k-1}}, \Phi_{k, \delta\Omega_k}, \Phi_{k+1, \Omega_{k+1}})$ satisfying the inequalities (192) but with $\phi_{k, \Omega(\square)}$ replaced by $\phi'_{k, \Omega(\square)}$ defined in (175). For $X \subset \delta\Omega_k$ define

$$\mathcal{P}'_k(X, \delta) = \bigcap_{\square \subset X} \mathcal{P}'_k(\square, \delta) \quad (195)$$

For all fields at once a natural domain might be $\cap_{j=1}^{k-1} [\mathcal{P}'_j(\delta\Omega_j, \delta)]_{L^{-(k-j)}} \cap \mathcal{P}_j(\Omega_k, \delta)$. Here the subscript $[\dots]_{L^{-(k-j)}}$ denotes that the indicated function of fields on \mathbb{T}_{M+N-j}^{-j} is to be scaled down to a function of fields on \mathbb{T}_{M+N-k}^{-k} . However we find that in the final region we need tighter restrictions near the boundaries relative to the bulk. So instead we take the definition:

Definition 3.4.

$$\mathcal{P}_{k, \Omega} = \bigcap_{j=1}^{k-1} \left[\mathcal{P}'_j(\delta\Omega_j, \delta) \right]_{L^{-(k-j)}} \cap \mathcal{P}_k(\Omega_k - \Omega_k^{2\mathfrak{q}}, \delta) \cap \mathcal{P}_k(\Omega_k^{2\mathfrak{q}}, 2\delta) \quad (196)$$

We also define a domain of analyticity for fields on the fine lattice $\mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k}$, as in part I.

Definition 3.5. Let $\epsilon > 2\delta$ be a fixed small positive number and $X \subset \mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k}$. Then $\mathcal{R}_k(X)$ is all functions $\phi : X \rightarrow \mathbb{C}$ such that :

$$\begin{aligned} |\phi| &< \lambda_k^{-1/4-3\epsilon} \\ |\partial\phi| &< \lambda_k^{-1/4-2\epsilon} \\ |\delta_\alpha \partial\phi| &< \lambda_k^{-1/4-\epsilon} \end{aligned} \tag{197}$$

3.3 the main theorem

We repeatedly block average starting with ρ_0 given by (3). Given $\rho_k(\Phi_k)$ we define first

$$\tilde{\rho}_{k+1}(\Phi_{k+1}) = \mathcal{N}_{aL, \mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^1}^{-1} \int \exp\left(-\frac{1}{2}aL|\Phi_{k+1} - Q\Phi_k|^2\right) \rho_k(\Phi_k) d\Phi_k \tag{198}$$

and then scale by

$$\rho_{k+1}(\Phi_{k+1}) = \tilde{\rho}_{k+1}(\Phi_{k+1,L}) L^{-|\mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^1|/2} \tag{199}$$

The theorem will assert that after k steps the density can be represented in the form

$$\begin{aligned} \rho_k(\Phi_k) = & Z_k \sum_{\mathbf{\Pi}} \int d\Phi_{k,\mathbf{\Omega}^c} dW_{k,\mathbf{\Pi}} K_{k,\mathbf{\Pi}} \mathcal{C}_{k,\mathbf{\Pi}} \\ & \chi_k(\Lambda_k) \exp\left(-S_k^+(\Lambda_k) + E_k(\Lambda_k) + R_{k,\mathbf{\Pi}}(\Lambda_k) + B_{k,\mathbf{\Pi}}(\Lambda_k)\right) \end{aligned} \tag{200}$$

where

$$\begin{aligned} d\Phi_{k,\mathbf{\Omega}^c} &= \prod_{j=0}^{k-1} \exp\left(-\frac{1}{2}aL^{-(k-j-1)}|\Phi_{j+1} - Q\Phi_j|_{\Omega_{j+1}^c}^2\right) d\Phi_{j,\Omega_{j+1}^c}^{(k-j)} \\ dW_{k,\mathbf{\Pi}} &= \prod_{j=0}^{k-1} (2\pi)^{-|\Omega_{j+1}-\Lambda_{j+1}|^{(j)}/2} \exp\left(-\frac{1}{2}L^{-(k-j)}|W_j|_{\Omega_{j+1}-\Lambda_{j+1}}^2\right) dW_{j,\Omega_{j+1}-\Lambda_{j+1}}^{(k-j)} \\ K_{k,\mathbf{\Pi}} &= \prod_{j=0}^k \exp\left(c_j|\Omega_j^{c,(j-1)}| - S_{j,L^{-(k-j)}}^{+,u}(\Lambda_{j-1} - \Lambda_j) + \left(\tilde{B}_{j,L^{-(k-j)}}\right)_{\mathbf{\Pi}_j}(\Lambda_{j-1}, \Lambda_j)\right) \\ \mathcal{C}_{k,\mathbf{\Pi}} &= \prod_{j=0}^k \left(\mathcal{C}_{j,L^{-(k-j)}}\right)_{\Lambda_{j-1}, \Omega_j, \Lambda_j} \end{aligned} \tag{201}$$

Here for $\Phi_{j+1,\Omega_j^c} : [\Omega_{j+1}^c]^{(j)} \rightarrow \mathbb{R}$ we define

$$d\Phi_{j,\Omega_{j+1}^c}^{(k-j)} = [L^{-(k-j)/2}]^{|\Omega_{j+1}^c|^{(j)}} \prod_{x \in [\Omega_{j+1}^c]^{(j)}} d\Phi_j(x) \tag{202}$$

Besides our basic variables $\Phi_{k,\mathbf{\Omega}}$ we also employ auxiliary variables

$$W_{k,\mathbf{\Pi}} = (W_{0,\Omega_1-\Lambda_1}, \dots, W_{k-1,\Omega_k-\Lambda_k}) \tag{203}$$

with $W_{j,\Omega_{j+1}-\Lambda_{j+1}} : [\Omega_{j+1}-\Lambda_{j+1}]^{(j)} \rightarrow \mathbb{R}$. The measure $dW_{j,\Omega_{j+1}-\Lambda_{j+1}}^{(k-j)}$ is defined as in (202).

We employ the convention that Λ_{-1}, Ω_0 are the full torus $\mathbb{T}_{\mathbf{M}+\mathbf{N}-k}^{-k}$.

Now we can state the main result:

Theorem 3.1. *Let L be sufficiently large, let M be sufficiently large (depending on L), let λ_k be sufficiently small (depending on L, M), and let $\bar{\mu}_k \leq \mathcal{O}(1)\lambda_k^{\frac{1}{4}}$. Let $\varepsilon_k, \lambda_k, \mu_k$ be the coupling constants selected in part I. Then the representation (200), (201) holds with the following properties:*

1. Z_k is the global normalization factor of part I. It satisfies $Z_0 = 1$ and

$$Z_{k+1} = Z_k \mathcal{N}_{a, \mathbb{T}_{M+N-k}^1}^{-1} (2\pi)^{|\mathbb{T}_{M+N-k}^0|/2} (\det C_k)^{1/2} \quad (204)$$

2. The characteristic function $\chi_k(\Lambda_k)$ forces the field Φ_k to be in the space $\mathcal{S}_k(\Lambda_k)$ defined in (177), (178). It has the form

$$\chi_k(\Lambda_k) = \prod_{\square \subset \Lambda_k} \chi_k(\square) \quad \chi_k(\square) = \chi(\Phi_k \in \mathcal{S}_k(\square)) \quad (205)$$

3. $\mathcal{C}_{k, \Lambda_{k-1}, \Omega_k, \Lambda_k}(\Phi_{k-1}, W_{k-1}, \Phi_k)$ is a collection of characteristic functions forcing certain fields to be large or small or both. The exact definition will be given in the course of the proof. It does not depend on Φ_{k, Λ_k^*} and enforces on $\Lambda_{k-1} - \Omega_k$

$$|\Phi_{k-1}| \leq 2p_{k-1} \lambda_{k-1}^{-\frac{1}{4}} L^{\frac{1}{2}} \quad |\partial \Phi_{k-1}| \leq 3p_{k-1} L^{\frac{3}{2}} \quad (206)$$

and enforces on $\Omega_k - \Lambda_k$ for some constant C_w

$$|\Phi_k| \leq 3p_{k-1} \lambda_{k-1}^{-1/4} L^{\frac{1}{2}} \quad |\partial \Phi_k| \leq 4p_{k-1} L^{\frac{3}{2}} \quad |W_{k-1}| \leq C_w p_{k-1} L^{\frac{1}{2}} \quad (207)$$

In the expression (201) this is scaled down to

$$\left(\mathcal{C}_{j, L^{-(k-j)}} \right)_{\Lambda_{j-1}, \Omega_j, \Lambda_j}(\Phi_{j-1}, W_{j-1}, \Phi_j) \equiv \mathcal{C}_{j, L^{k-j}(\Lambda_{j-1}, \Omega_j, \Lambda_j)}(\Phi_{j-1, L^{k-j}}, W_{j, L^{k-j}}, \Phi_{j, L^{k-j}}) \quad (208)$$

4. The bare action is $S_k^+(\Lambda_k) = S_k^+(\Lambda_k, \Phi_k, \phi_k, \Omega(\Lambda_k^*))$ where $\phi_k, \Omega(\Lambda_k^*)$ is defined in (170) and

$$\begin{aligned} S_k^+(\Lambda_k, \Phi_k, \phi) &= S_k^*(\Lambda_k, \Phi_k, \phi) + V_k(\Lambda_k, \phi) \\ S_k^*(\Lambda_k, \Phi_k, \phi) &= \frac{a_k}{2} \|\Phi_k - Q_k \phi\|_{\Lambda_k}^2 + \frac{1}{2} \|\partial \phi\|_{*, \Lambda_k}^2 + \frac{1}{2} \bar{\mu}_k \|\phi\|_{\Lambda_k}^2 \\ V_k(\Lambda_k, \phi) &= \varepsilon_k \text{Vol}(\Lambda_k) + \frac{1}{2} \mu_k \|\phi^2\|_{\Lambda_k} + \frac{1}{4} \lambda_k \int_{\Lambda_k} \phi^4 \end{aligned} \quad (209)$$

5. $E_k(\Lambda_k) = E_k(\Lambda_k, \phi_k, \Omega(\Lambda_k^*))$ are the main corrections to the bare action and have the local structure

$$E_k(\Lambda_k) = \sum_{X \in \mathcal{D}_k, X \subset \Lambda_k} E_k(X) \quad (210)$$

The functionals $E_k(X, \phi_k, \Omega(\Lambda_k^*))$ are restrictions of functionals $E_k(X, \phi)$ analytic in $\phi \in \mathcal{R}_k(\Lambda_k)$ and satisfying there for $\beta < \frac{1}{4} - 10\epsilon$

$$|E_k(X)| \leq \lambda_k^\beta e^{-\kappa d_M(X)} \quad (211)$$

They are identical with the global small field functions $E_k(X, \phi)$ of part I.

6. $R_{k,\mathbf{\Pi}}(\Lambda_k) = R_{k,\mathbf{\Pi}}(\Lambda_k, \Phi_k)$ is a tiny remainder and has the local structure

$$R_{k,\mathbf{\Pi}}(\Lambda_k) = \sum_{X \in \mathcal{D}_k, X \subset \Lambda_k} R_{k,\mathbf{\Pi}}(X) \quad (212)$$

The function $R_{k,\mathbf{\Pi}}(X, \Phi_k)$ is analytic in $\mathcal{P}_k(\Lambda_k, 2\delta)$, and satisfies there for a fixed integer $n_0 \geq 4$:

$$|R_{k,\mathbf{\Pi}}(X)| \leq \lambda_k^{n_0} e^{-\kappa d_M(X)} \quad (213)$$

7. The active boundary term has the form $B_{k,\mathbf{\Pi}}(\Lambda_k) = B_{k,\mathbf{\Pi}}(\Lambda_k; \Phi_k, \mathbf{\Omega}, W_{k,\mathbf{\Pi}})$. It has the local expansion

$$B_{k,\mathbf{\Pi}}(\Lambda_k) = \sum_{X \in \mathcal{D}_k(\text{mod } \Omega_k^c), X \# \Lambda_k, X \subset \Omega_1} B_{k,\mathbf{\Pi}}(X) \quad (214)$$

The function $B_{k,\mathbf{\Pi}}(X, \Phi_k, \mathbf{\Omega}, W_{k,\mathbf{\Pi}})$ is analytic in $\mathcal{P}_{k,\mathbf{\Omega}}$ and for some $B_w > C_w$

$$|W_j| \leq B_w p_j L^{\frac{1}{2}(k-j)} \quad \text{on } \Omega_{j+1} - \Lambda_{j+1} \quad (215)$$

and satisfies there

$$|B_{k,\mathbf{\Pi}}(X)| \leq B_0 \lambda_k^\beta e^{-\kappa d_M(X, \text{mod } \Omega_k^c)} \quad (216)$$

for some constant B_0 depending on L, M .

8. The inactive boundary term has the form $\tilde{B}_{k,\mathbf{\Pi}}(\Lambda_{k-1}, \Lambda_k, \Phi_k, \mathbf{\Omega}_k, W_{k,\mathbf{\Pi}})$. It depends on the variables only in $\Omega_1 - \Lambda_k$, is analytic in $\mathcal{P}_{k,\mathbf{\Omega}}$ and (215) and satisfies there

$$|\tilde{B}_{k,\mathbf{\Pi}}(\Lambda_{k-1}, \Lambda_k)| \leq B_0 \left| \bar{\Lambda}_{k-1}^{(k)} - \Lambda_k^{(k)} \right| \quad (217)$$

In the expression (201) we have the scaled version for $j \leq k$

$$\begin{aligned} & \left(\tilde{B}_{j,L^{-(k-j)}} \right)_{\mathbf{\Pi}_j}(\Lambda_{j-1}, \Lambda_j; \Phi_j, \mathbf{\Omega}_j, W_{j,\mathbf{\Pi}_j}) \\ & \equiv \tilde{B}_{j,L^{k-j}} \mathbf{\Pi}_j \left(L^{k-j} \Lambda_{j-1}, L^{k-j} \Lambda_j; (\Phi_j, \mathbf{\Omega}_j)_{L^{k-j}}, (W_{j,\mathbf{\Pi}_j})_{L^{k-j}} \right) \end{aligned} \quad (218)$$

where $\mathbf{\Omega}_j, \mathbf{\Pi}_j$ are $\mathbf{\Omega}, \mathbf{\Pi}$ truncated at the j^{th} level.

9. With $\delta\Lambda_{k-1} = \Lambda_{k-1} - \Lambda_k$, the unrenormalized action is $S_k^{+,u}(\delta\Lambda_{k-1}, \Phi_k, \mathbf{\Omega}, \phi_k, \mathbf{\Omega}(\Lambda_{k-1}, \Omega_k, \Lambda_k))$ where $\phi_k, \mathbf{\Omega}(\Lambda_{k-1}, \Omega_k, \Lambda_k)$ is defined in (172) and

$$\begin{aligned} S_k^{+,u}(\delta\Lambda_{k-1}, \Phi_k, \mathbf{\Omega}, \phi) &= S_k^*(\delta\Lambda_{k-1}, \Phi_k, \mathbf{\Omega}, \phi) + V_k^u(\delta\Lambda_{k-1}, \phi) \\ V_k^u(\Lambda, \phi) &= L^3 \varepsilon_{k-1} \text{Vol}(\Lambda) + \frac{1}{2} L^2 \mu_{k-1} \|\phi^2\|_\Lambda + \frac{1}{4} \lambda_k \int_\Lambda \phi^4 \end{aligned} \quad (219)$$

In the expression (201) we have the scaled version for $j \leq k$

$$\begin{aligned} & S_{j,L^{-(k-j)}}^{+,u}(\delta\Lambda_{j-1}, \Phi_j, \mathbf{\Omega}_j, \phi_j, \mathbf{\Omega}(\delta\Lambda_{j-1}^*)) \\ &= S_j^{+,u} \left(L^{k-j}(\delta\Lambda_{j-1}), \Phi_j, \mathbf{\Omega}_j, L^{k-j}, \phi_j, \mathbf{\Omega}(\delta\Lambda_{j-1}^*), L^{k-j} \right) \end{aligned} \quad (220)$$

Remarks.

1. The functions $S_k^+(\Lambda_k)$, $E_k(\Lambda_k)$, $R_{k,\mathbf{\Pi}}(\Lambda_k)$, and $B_{k,\mathbf{\Pi}}(\Lambda_k)$ depend on fields in the current small field region Λ_k and contribute to the next RG transformation. The functions $S_j^+(\Lambda_{j-1} - \Lambda_j)$ and $\tilde{B}_{j,\mathbf{\Pi}_j}(\Lambda_{j-1}, \Lambda_j)$ do not depend on fields in Λ_k and do not contribute to the next RG transformation.
2. The conditions that λ_k be small and that $\bar{\mu}_k \leq \mathcal{O}(1)\lambda_k^{\frac{1}{4}}$ are a constraint on how long we can iterate the procedure, not on the bare parameters $\lambda, \bar{\mu}$ which are unrestricted.

The statement that the coupling constants $\varepsilon_k, \lambda_k, \mu_k$ in V_k are chosen as in part I means that they satisfy discrete dynamical equations

$$\begin{aligned}\varepsilon_{k+1} &= L^3 \varepsilon_k + \mathcal{L}_1 E_k + \varepsilon_k^*(\lambda_k, \mu_k, E_k) \\ \mu_{k+1} &= L^2 \mu_k + \mathcal{L}_2 E_k + \mu_k^*(\lambda_k, \mu_k, E_k) \\ \lambda_{k+1} &= L \lambda_k \\ E_{k+1} &= \mathcal{L}_3 E_k + E_k^*(\lambda_k, \mu_k, E_k)\end{aligned}\tag{221}$$

and they are tuned so that $|\varepsilon_k| \leq \mathcal{O}(1)\lambda_k^\beta$ and $|\mu_k| \leq \lambda_k^{\frac{1}{2}+\beta}$.

The unrenormalized potential V_j^u differs from the renormalized potential V_j only in that the last corrections to the coupling constants are not included. That is we have energy density $L^3 \varepsilon_{k-1}$ instead of $\varepsilon_k = L^3 \varepsilon_{k-1} + \dots$ and mass $L^2 \mu_{k-1}$ instead of $\mu_k = L^2 \mu_{k-1} + \dots$.

3. With more work we could probably identify the history dependent parts of $R_{k,\mathbf{\Pi}}$ as boundary terms and so get a new R_k independent of $\mathbf{\Pi}$. We note that in Balaban's models these R_k terms get additional contributions from a recycling operation that converts some large field contributions back to small field contributions (the "R - operation"). This difficult step is not necessary for this model.

To check that our formula makes sense we need the following:

Lemma 3.3. *The bounds of the characteristic functions $\mathcal{C}_{k,\mathbf{\Pi}}$ and $\chi_k(\Lambda_k)$ put the various fields in the analyticity domains for $E_k(\Lambda_k)$, $R_{k,\mathbf{\Pi}}(\Lambda_k)$, $B_{k,\mathbf{\Pi}}(\Lambda_k)$, $\tilde{B}_{j,\mathbf{\Pi}_j}(\Lambda_{j-1}, \Lambda_j)$*

Proof. First note that they imply

$$\begin{aligned}|\Phi_k| &\leq 3p_{k-1}\lambda_{k-1}^{-\frac{1}{4}}L^{\frac{1}{2}} && \text{on } \Omega_k \\ |\Phi_j| &\leq 3p_{j-1}\lambda_{j-1}^{-1/4}L^{\frac{1}{2}(k-j)} && \text{on } \Omega_j - \Omega_{j+1} \quad j = 1, \dots, k-1 \\ |\Phi_0| &\leq 2p_0\lambda_0^{-1/4}L^{\frac{1}{2}k} && \text{on } \Omega_0 - \Omega_1\end{aligned}\tag{222}$$

These all are enforced by $\mathcal{C}_{k,\mathbf{\Pi}}$ except that it only gives that first on $\Omega_k - \Lambda_k$. By lemma 3.1 the characteristic function $\chi_k(\Lambda_k)$ gives a stronger bound on Λ_k .

The bound on Φ_k implies $|\tilde{Q}_{k,\mathbb{T}^0,\mathbf{\Omega}(\Lambda_k^*)}^T \Phi_k| \leq 3p_{k-1}\lambda_{k-1}^{-\frac{1}{4}}L^{\frac{1}{2}}$ on a neighborhood of $\Omega_1(\Lambda_k^*)$, since the latter is contained in Ω_k . Then (133) gives that on Λ_k

$$|\phi_{k,\mathbf{\Omega}(\Lambda_k^*)}|, \quad |\partial\phi_{k,\mathbf{\Omega}(\Lambda_k^*)}|, \quad |\delta_\alpha\partial\phi_{k,\mathbf{\Omega}(\Lambda_k^*)}| \leq Cp_{k-1}\lambda_{k-1}^{-1/4}\tag{223}$$

But for λ_k sufficiently small $Cp_{k-1}\lambda_{k-1}^{-1/4} \leq \lambda_k^{-\frac{1}{4}-\epsilon}$ so $\phi_{k,\mathbf{\Omega}(\Lambda_k^*)} \in \mathcal{R}_k(X)$ for $X \subset \Lambda_k$. Thus we are in the analyticity region for $E_k(\Lambda_k)$.

Similarly for λ_k sufficiently small $|\Phi_k| \leq 3p_{k-1}\lambda_{k-1}^{-\frac{1}{4}}$ implies for $\square \subset \Lambda_k$ on $\tilde{\square}$

$$|\Phi_k - Q_k \phi_{k,\Omega(\square)}|, \quad |\partial \phi_{k,\Omega(\square)}|, \quad |\phi_{k,\Omega(\square)}| \leq Cp_{k-1}\lambda_{k-1}^{-1/4} \leq \lambda_k^{-\frac{1}{4}-\delta} \quad (224)$$

Thus we are in $\mathcal{P}_k(\square)$ and hence in $\mathcal{P}_k(\Lambda_k)$ which is the domain for $R_{k,\Pi}(\Lambda_k)$.

Next note that (222) implies that $|\Phi_{j,L^{k-j}}| \leq 3p_{j-1}\lambda_{j-1}^{-1/4}$ on $L^{k-j}\delta\Omega_j$. Then for an M -cube \square well inside $L^{k-j}\delta\Omega_j$ we have on $\tilde{\square}$

$$|\Phi_{j,L^{k-j}} - Q_j \phi_{j,\Omega(\square)}(\Phi_{j,L^{k-j}})|, \quad |\partial \phi_{j,\Omega(\square)}(\Phi_{j,L^{k-j}})|, \quad |\phi_{j,\Omega(\square)}(\Phi_{j,L^{k-j}})| \leq Cp_{j-1}\lambda_{j-1}^{-1/4} \leq \lambda_j^{-\frac{1}{4}-\delta} \quad (225)$$

The bound says for $j = k$ that $\Phi_k \in \mathcal{P}_k(\square, \delta)$, and for $j < k$ that $\Phi_{j,L^{k-j}} \in \mathcal{P}'_j(\square, \delta)$ or $\Phi_j \in [\mathcal{P}'_j(\square, \delta)]_{L^{-(k-j)}}$. The same conclusion holds for any $\square \subset L^{k-j}\delta\Omega_j$, but now involves fields from adjacent regions. Thus we are in $\mathcal{P}_{k,\Omega}$. Also since $C_w < B_w$ we have that $|W_j| \leq C_w p_j L^{\frac{1}{2}(k-j)}$ implies $|W_j| \leq B_w p_j L^{\frac{1}{2}(k-j)}$. Thus we are in the analyticity domain for $B_{k,\Pi}(\Lambda_k)$.

The analysis for $\tilde{B}_{j,\Pi_j}(\Lambda_{j-1}, \Lambda_j)$ is similar.

3.4 initial representation

We begin the proof of theorem 3.1 by showing that the representation holds for $k = 0$. We have initially

$$\rho_0(\Phi_0) = \exp\left(-S_0^+(\Phi_0)\right) = \exp\left(-\frac{1}{2}\|\partial\Phi_0\|^2 - \frac{1}{2}\bar{\mu}_0\|\Phi_0\|^2 - V_0(\Phi_0)\right) \quad (226)$$

We break into large and small field regions as follows. For each M -cube \square define characteristic functions by

$$\chi_0(\square, \Phi_0) = \prod_{x \in \square} \chi\left(|\partial\Phi_0(x)| \leq p_0, |\Phi_0(x)| \leq \lambda_0^{-1/4}p_0\right) \quad (227)$$

Then we write with $\zeta_0(\square) = 1 - \chi_0(\square)$ and with Q_0 a union of M cubes

$$1 = \sum_{\square \subset \mathbb{T}_{M+N}^0} (\zeta_0(\square) + \chi_0(\square)) = \sum_{Q_0} \prod_{\square \subset Q_0} \zeta_0(\square) \prod_{\square \subset Q_0^c} \chi_0(\square) \equiv \sum_{Q_0} \zeta_0(Q_0) \chi_0(Q_0^c) \quad (228)$$

Then in Q_0^c the inequalities $|\partial\Phi_0| \leq p_0, |\Phi_0| \leq \lambda_0^{-1/4}p_0$ hold at every point, whereas in every cube in Q_0 some inequality is violated at some point.

Now Q_0^c is an adequate small field region, but for consistency with subsequent steps we shrink it. Let $\Lambda_0 = (Q_0^c)^{5\sharp}$ or $\Lambda_0^c = Q_0^{5*}$ and rewrite (228) as

$$1 = \sum_{\Lambda_0} \mathcal{C}_{0,\Lambda_0} \chi_0(\Lambda_0) \quad \text{where} \quad \mathcal{C}_{0,\Lambda_0} = \sum_{Q_0: Q_0^{4*} = \Lambda_0^c} \zeta_0(Q_0) \chi_0(Q_0^c - \Lambda_0) \quad (229)$$

Juxtaposing this with ρ_0 and splitting the action on Λ_0 we have

$$\rho_0 = \sum_{\Lambda_0} \mathcal{C}_{0,\Lambda_0} \exp\left(-S_0^+(\Lambda_0)\right) \chi_0(\Lambda_0) \exp\left(-S_0^+(\Lambda_0)\right) \quad (230)$$

This is the representation (200) if we make the following interpretations. There are no integrals, $Z_0 = 1$, and $\Lambda_{-1} = \Omega_0 = \mathbb{T}_{M+N}^0$. The functions $E_0, R_0, B_0, \tilde{B}_0$ are all zero. The as yet undefined fields $\phi_{0,\Omega(\Lambda_0^*)}$ and $\phi_{0,\Omega((\Lambda_0^c)^*)}$ are just Φ_0 and $S_0^{+,u}(\Lambda_0^c) = S_0^+(\Lambda_0^c)$. We also interpret $\phi_{0,\Omega(\square)}$ as Φ_0 and Q_0 as the identity, and then $\chi_0(\square)$ defined in (227) is the same as (205).

3.5 new small field region

We have seen that theorem 3.1 is true for $k = 0$. To complete the proof we assume it is true for k and generate the representation for $k + 1$. This will occupy the rest of the paper. The proof follows especially [8]- [14].

To begin insert the expression (200) for ρ_k into the definition (198) of ρ_{k+1} , and bring the sums outside the integral.

In the current small field region Λ_k we have limits on the size of Φ_k . Because of the factor $\exp(-\frac{1}{2}aL|\Phi_{k+1} - Q\Phi_k|^2)$ large Φ_{k+1} will also be suppressed in Λ_k . To take advantage of this feature we proceed as follows. For any LM -cube \square in \mathbb{T}_{M+N-k}^{-k} define the characteristic function

$$\chi_k^q(\square, \Phi_{k+1}, \Phi_k) = \prod_{y \in \square} \chi(|\Phi_{k+1}(y) - (Q\Phi_k)(y)| \leq p_k) \quad (231)$$

Also let $\bar{\Lambda}_k$ be the union of all LM cubes intersecting Λ_k . Then with $\zeta_k^q(\square) = 1 - \chi_k^q(\square)$

$$\begin{aligned} 1 &= \prod_{\square \subset \bar{\Lambda}_k} \zeta_k^q(\square) + \chi_k^q(\square) \\ &= \sum_{P_{k+1} \subset \bar{\Lambda}_k} \prod_{\square \subset P_{k+1}} \zeta_k^q(\square) \prod_{\square \subset \bar{\Lambda}_k - P_{k+1}} \chi_k^q(\square) \\ &\equiv \sum_{P_{k+1} \subset \bar{\Lambda}_k} \zeta_k^q(P_{k+1}) \chi_k^q(\bar{\Lambda}_k - P_{k+1}) \end{aligned} \quad (232)$$

where P_{k+1} is a union of LM -cubes. Now given $\bar{\Lambda}_k$ and P_{k+1} define a new small field region Ω_{k+1} by (See figure 5)

$$\Omega_{k+1} = (\bar{\Lambda}_k)^{5\natural} - P_{k+1}^{5*} \quad \text{or} \quad \Omega_{k+1}^c = (\bar{\Lambda}_k)^{c,5*} \cup P_{k+1}^{5*} \quad (233)$$

Here the $*$, \natural operations refer to adding or deleting layers of LM -cubes. In generating Ω_{k+1}^c from $(\bar{\Lambda}_k)^c$ we add at least $5[r_{k+1}]$ layers of LM -cubes so $d((\bar{\Lambda}_k)^c, \Omega_{k+1}) \geq 5[r_{k+1}]LM$. This is the required separation at this scale.

Now classify the terms in the sum by the union of LM -cubes Ω_{k+1} that they generate and find

$$1 = \sum_{\Omega_{k+1} \subset \bar{\Lambda}_k^{5\natural}} \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) \chi_k^q(\Omega_{k+1}) \quad (234)$$

where

$$\mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) = \sum_{P_{k+1} \subset \bar{\Lambda}_k : \Omega_{k+1} = (\bar{\Lambda}_k)^{5\natural} - P_{k+1}^{5*}} \zeta_k^q(P_{k+1}) \chi_k^q((\bar{\Lambda}_k - P_{k+1}) - \Omega_{k+1}) \quad (235)$$

We have

$$|\Phi_{k+1} - Q\Phi_k| \leq p_k \quad \text{on} \quad \Omega_{k+1} \quad (236)$$

Insert (234) under the integral sign in our expression. Split the integral over Φ_k into an integral over Φ_k, Ω_{k+1}^c and an integral over Φ_k, Ω_{k+1} . We define with

$$\Omega^+ = (\Omega, \Omega_{k+1}) = (\Omega_1, \dots, \Omega_{k+1}) \quad (237)$$

the measure

$$d\Phi_{k+1}^0, \Omega^{+,c} = \exp\left(-\frac{aL}{2}|\Phi_{k+1} - Q\Phi_k|_{\Omega_{k+1}^c}^2\right) d\Phi_{k, \Omega_{k+1}^c} d\Phi_{k, \Omega^c} \quad (238)$$

We also transfer the potential from $S^+(\Lambda_k)$ to $E(\Lambda_k)$ writing $-S^+(\Lambda_k) + E(\Lambda_k) = -S^*(\Lambda_k) + E^+(\Lambda_k)$ where

$$E_k^+(\Lambda_k) = E_k(\Lambda_k) - V_k(\Lambda_k) \quad (239)$$

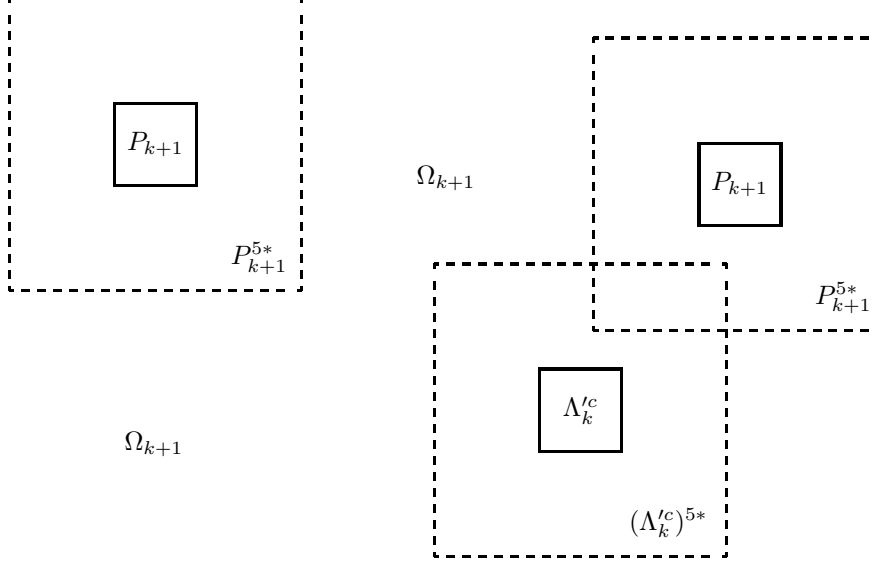


Figure 5: Illustrating $\Omega_{k+1}^c = (\bar{\Lambda}_k)^{c,5*} \cup P_{k+1}^{5*}$. Here Ω_{k+1}^c is the region inside the dotted lines.

Finally moving the integral over $\Phi_{k,\Omega_{k+1}}$ inside and taking account that $K_{k,\mathbf{\Pi}}, \mathcal{C}_{k,\mathbf{\Pi}}, \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1})$ do not depend on $\Phi_{k,\Omega_{k+1}}$ we have the expression:

$$\begin{aligned} \tilde{\rho}_{k+1}(\Phi_{k+1}) = & Z_k \mathcal{N}_{aL, \mathbb{T}_{M+N-k}^1}^{-1} \sum_{\mathbf{\Pi}, \Omega_{k+1}} \int d\Phi_{k+1, \Omega_{k+1}}^0 dW_{k, \mathbf{\Pi}} K_{k, \mathbf{\Pi}} \mathcal{C}_{k, \mathbf{\Pi}} \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) \\ & \int d\Phi_{k, \Omega_{k+1}} \exp\left(-\frac{1}{2}aL|\Phi_{k+1, L} - Q\Phi_k|_{\Omega_{k+1}}^2\right) \\ & \chi_k(\Lambda_k) \chi_k^q(\Omega_{k+1}) \exp\left(-S_k^*(\Lambda_k) + E_k^+(\Lambda_k) + R_{k, \mathbf{\Pi}}(\Lambda_k) + B_{k, \mathbf{\Pi}}(\Lambda_k)\right) \end{aligned} \quad (240)$$

Let us collect the bounds implied by the characteristic functions $\mathcal{C}_{k, \mathbf{\Pi}} \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) \chi_k(\Lambda_k) \chi_k^q(\Omega_{k+1})$.

Lemma 3.4. *The characteristic functions enforce the following bounds:*

$$|\Phi_k| \leq 3p_{k-1} \lambda_{k-1}^{-1/4} L^{\frac{1}{2}} \quad |\partial\Phi_k| \leq 4p_{k-1} L^{\frac{3}{2}} \quad \text{on } \Omega_k - \Lambda_k \quad (241)$$

$$|\Phi_k| \leq 2p_k \lambda_k^{-1/4} \quad |\partial\Phi_k| \leq 3p_k \quad \text{on } \tilde{\Lambda}_k \quad (242)$$

$$|\Phi_{k+1}| \leq 3p_k \lambda_k^{-1/4} \quad |\partial\Phi_{k+1}| \leq 4p_k \quad \text{on } \Omega_{k+1} \quad (243)$$

In addition $\Phi_{k+1}^\# = (Q\Phi_{k, \delta\Omega_k}, \Phi_{k+1, \Omega_{k+1}})$ satisfies

$$|\Phi_{k+1}^\#| \leq Cp_k \lambda_k^{-1/4} \quad |\partial\Phi_{k+1}^\#| \leq Cp_k \quad \text{on } \Omega_k \quad (244)$$

Proof. The characteristic function $\mathcal{C}_{k, \mathbf{\Pi}}$ enforces the bounds (241) on $\Omega_k - \Lambda_k$ by assumption. The function $\chi_k(\Lambda_k)$ says that for $\square \subset \Lambda_k$ we have $\Phi_k \in \mathcal{S}_k(\square)$. Then by lemma 3.1 we have on $\tilde{\square}$ the bounds $|\Phi_k| \leq 2p_k \lambda_k^{-\frac{1}{4}}$ and $|\partial\Phi_k| \leq 3p_k$. This gives (242).

The first bound in (242) and (236) give $|\Phi_{k+1}| \leq 3p_k \lambda_k^{-\frac{1}{4}}$ on Ω_{k+1} . The second bound follows by (236) as well by the estimate for $y, y + Le_\mu \in \Omega_k^{(k+1)}$

$$\begin{aligned} |\partial_\mu \Phi_{k+1}(y)| &= L^{-1} |\Phi_{k+1}(y + Le_\mu) - \Phi_{k+1}(y)| \\ &\leq L^{-1} |(Q\Phi_k)(y + Le_\mu) - Q\Phi_k(y)| + L^{-1} 2p_k \leq 3p_k + L^{-1} 2p_k \leq 4p_k \end{aligned} \quad (245)$$

Thus (243) is established.

For (244) note that (241), (242) imply that $|\Phi_k| \leq Cp_k \lambda_k^{-\frac{1}{4}}$ and $|\partial\Phi_k| \leq Cp_k$ on Ω_k . Hence $Q\Phi_k$ satisfies the same bounds on $\delta\Omega_k$, as does Φ_{k+1} on Ω_{k+1} . The remaining issue is when a derivative crosses the boundary of Ω_{k+1} . This is handled as follows. Suppose $y \in \Omega_k^{(k+1)}$ and $y + Le_\mu \in \Omega_{k+1}^{(k+1)}$. Then

$$\begin{aligned} (\partial_\mu \Phi_{k+1}^\#)(y) &= L^{-1} \left(\Phi_{k+1}(y + Le_\mu) - (Q\Phi_k)(y) \right) \\ &= L^{-1} \left(\Phi_{k+1}(y + Le_\mu) - (Q\Phi_k)(y + Le_\mu) \right) + L^{-1} \left((Q\Phi_k)(y + Le_\mu) - (Q\Phi_k)(y) \right) \end{aligned} \quad (246)$$

The first term is bounded by Cp_k by (236) and the second term is bounded by Cp_k by the bound on $\partial\Phi_k$. This completes the proof.

3.6 an approximate minimizer

The two quadratic terms in the exponents in (240) can be identified as

$$J_{\Lambda_k, \Omega_{k+1}}^*(\Lambda_k, \Phi_{k+1}, \Phi_k, \phi_{\Omega(\Lambda_k^*)}) = \frac{1}{2} \frac{a}{L^2} \|\Phi_{k+1} - Q\Phi_k\|_{\Omega_{k+1}}^2 + S_k^*(\Lambda_k, \Phi_k, \phi_{\Omega(\Lambda_k^*)}) \quad (247)$$

defined previously in (87).

We want to find the minimizer of this functional in the variable in $\Phi_{k, \Omega_{k+1}}$. This is not the standard setup because the action is restricted to Λ_k , but it is a problem we have anticipated. Instead we use an approximate minimizer, namely the minimizer for the full problem on

$$\Omega' \equiv (\Omega(\Lambda_k^*), \Omega_{k+1}) \quad (248)$$

By lemma 2.3 the minimum in $\Phi_{k, \Omega_{k+1}}$ for that problem comes at

$$\Psi_{k, \Omega_{k+1}}(\Omega') = Q_k \phi_{k+1, \Omega'}^0 - \frac{aL^{-2}}{a_k + aL^{-2}} Q^T Q_{k+1} \phi_{k+1, \Omega'}^0 + \frac{aL^{-2}}{a_k + aL^{-2}} Q^T \Phi_{k+1} \quad (249)$$

where $\phi_{k+1, \Omega'}^0$ is defined in (55). Recalling that $\phi_{k, \Omega(\Lambda_k^*)} = \phi_{k, \Omega(\Lambda_k^*)}(\tilde{Q}_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k)$ the minimizer in ϕ is more precisely characterized as $\phi_{k+1, \Omega'}^0 = \phi_{k+1, \Omega'}^0(\hat{\Phi}_{k+1, \Omega'})$ where

$$\hat{\Phi}_{k+1, \Omega'} = \left([\tilde{Q}_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k]_{\Omega_{k+1}^c}, \Phi_{k+1, \Omega_{k+1}} \right) \quad (250)$$

If $k = 0$ then the minimizer in Φ_{0, Ω_1} is just $\Psi_{0, \Omega_1} = \phi_{1, \Omega_1}^0$ as a separate calculation reveals.

Inserting $\Psi_{k, \Omega_{k+1}}(\Omega')$ into $\tilde{Q}_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k$ gives

$$\hat{\Psi}_{k, \Omega'} \equiv \left([\tilde{Q}_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k]_{\Omega_{k+1}^c}, \Psi_{k, \Omega_{k+1}}(\Omega') \right) \quad (251)$$

Note that $\hat{\Phi}_{k+1, \Omega'}$ and $\hat{\Psi}_{k, \Omega'}$ now include boundary fields in $\Omega_1(\Lambda_k^*)^c$. This is a different convention from chapter 2 where such fields were treated separately. The identity

$$\phi_{k+1, \Omega'}^0 = \phi_{k, \Omega(\Lambda_k^*)}(\hat{\Psi}_{k, \Omega'}) \quad (252)$$

holds by lemma 2.3. (The fields in Ω_{k+1}^c are just spectators in the proof of this identity.)

Now expand $J_{\Lambda_k, \Omega_{k+1}}^*(\Lambda_k)$ around the minimizer inserting $\Phi_{k, \Omega_{k+1}} = \Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}') + Z$. Using the identity (252) this entails

$$\begin{aligned}\hat{\Phi}_{k, \mathbf{\Omega}(\Lambda_k^*)} &= \hat{\Psi}_{k, \mathbf{\Omega}'} + (0, Z) \\ \phi_{k, \mathbf{\Omega}(\Lambda_k^*)} &= \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)}\end{aligned}\quad (253)$$

Here Z is a function on the unit lattice $\Omega_{k+1}^{(k)}$ and as before

$$\mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)} = \phi_{k, \mathbf{\Omega}(\Lambda_k^*)}(0, Z) = a_k G_{k, \mathbf{\Omega}(\Lambda_k^*)} Q_k^T Z \quad (254)$$

Lemma 2.5 is applicable here and so ⁶

$$\begin{aligned}J_{\Lambda_k, \Omega_{k+1}}^*(\Lambda_k, \Phi_{k+1}, \hat{\Psi}_{k, \mathbf{\Omega}'} + (0, Z), \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)}) &= S_{k+1}^{*,0}(\Lambda_k, \Phi_{k+1}, \mathbf{\Omega}^+, \phi_{k+1, \mathbf{\Omega}'}^0) \\ &+ \frac{1}{2} \left\langle Z, \left[\Delta_{k, \mathbf{\Omega}(\Lambda_k^*)} + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} Z \right\rangle + R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)}\end{aligned}\quad (255)$$

where

$$\begin{aligned}R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)} &= b_{\Lambda_k} \left[\partial \phi_{k+1, \mathbf{\Omega}'}^0, \mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)} \right] + \frac{1}{2} \|a^{1/2} Q_{k, \mathbf{\Omega}(\Lambda_k^*)} \mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)}\|_{\Lambda_k^c}^2 \\ &+ \frac{1}{2} \|\partial \mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)}\|_{*, \Lambda_k^c}^2 + \frac{1}{2} \bar{\mu}_k \|\mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)}\|_{\Lambda_k^c}^2\end{aligned}\quad (256)$$

The function $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)}$ is tiny because Z is localized in Ω_{k+1} , and $\mathcal{Z}_{k, \mathbf{\Omega}(\Lambda_k^*)}$ is evaluated on Λ_k^c , and the operator connecting these distant sets, namely $G_{k, \mathbf{\Omega}(\Lambda_k^*)}$, has an exponentially decaying kernel. Furthermore $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)}$ has a local expansion. However we postpone the demonstration of such facts to section 3.13.

The idea is now to make the substitutions (253) in the integral (240) and then integrate over Z instead of $\Phi_{k, \Omega_{k+1}}$, taking advantage of (255). This substitution is not completely satisfactory because when it appears in the characteristic function it will introduce non-local non-analytic dependence on Φ_{k+1} everywhere inside Λ_k . This we want to avoid. Instead we replace it by a more local version.

3.7 a better approximation

To develop the more local version we introduce some definitions. In $\phi_{k+1, \mathbf{\Omega}'}^0$ we have the propagator $G_{k+1, \mathbf{\Omega}'}^0$. A weakened propagator $G_{k+1, \mathbf{\Omega}'}^0(s)$ is defined as in section 2.5. Here $s = \{s_\square\}$ is a collection of variables indexed by cubes \square associated with $\mathbf{\Omega}'$, that is $\square \subset \delta\Omega_j'$ is an $L^{k-j}M$ cube, and in particular \square is an LM cube in Ω_{k+1} . This gives a more local field

$$\phi_{k+1, \mathbf{\Omega}'}^0(s) = G_{k+1, \mathbf{\Omega}'}^0(s) \left(L^{-2} Q_{k+1, \mathbf{\Omega}'}^T a^{(k+1)} [\hat{\Phi}_{k+1, \mathbf{\Omega}'}]_{\Omega_1'} + \Delta_{\Omega_1', \Omega_1^c} Q_k^T \Phi_k \right) \quad (257)$$

with $\hat{\Phi}_{k+1, \mathbf{\Omega}'}$ defined in (250). Then define a more local minimizer by

$$\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}', s) = Q_k \phi_{k+1, \mathbf{\Omega}'}^0(s) - \frac{aL^{-2}}{a_k + aL^{-2}} Q^T Q_{k+1} \phi_{k+1, \mathbf{\Omega}'}^0(s) + \frac{aL^{-2}}{a_k + aL^{-2}} Q^T \Phi_{k+1} \quad (258)$$

⁶ Because of the restriction to Λ_k the substitution $\hat{\Phi}_{k, \mathbf{\Omega}(\Lambda_k^*)} = \hat{\Psi}_{k, \mathbf{\Omega}'} + (0, Z)$ is the same as the expected $\Phi_k = (\Phi_{k, \Lambda_k - \Omega_{k+1}}, \Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}') + Z)$. The restriction also allows us to replace $\Phi_{k+1, \mathbf{\Omega}'}$ by $\Phi_{k+1, \mathbf{\Omega}^+}$ in $S_{k+1}^{*,0}(\Lambda_k)$.

Next let \square be an LM cube in Ω_{k+1} , and let \square^* be the same with $[r_{k+1}]$ layers of LM cubes added. We define

$$G_{k+1, \Omega'}^0(\square^*) = G_{k+1, \Omega'}^0(s_{\square^*} = 1, s_{\square^*, c} = 0) \quad (259)$$

which has no coupling outside of \square^* . This does not depend on the full extent of Ω' ; we could as well take Ω^+ here. Similarly define $\phi_{k+1, \Omega'}^0(\square^*)$ and $\Psi_{k, \Omega_{k+1}}(\Omega', \square^*)$. Then we define a more localized field $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega', x)$ to be equal to $\Psi_{k, \Omega_{k+1}}(\Omega', \square^*, x)$ for $x \in \Omega_{k+1}^{(k)} \cap \square$. This can also be written

$$\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega') = \sum_{\square \subset \Omega_{k+1}} 1_{\square} \Psi_{k, \Omega_{k+1}}(\Omega', \square^*) \quad (260)$$

with the spectator fields present this gives

$$\hat{\Psi}_{k, \Omega'}^{\text{loc}} \equiv ([\tilde{Q}_{\mathbb{T}^0, \Omega(\Lambda_k^*)}^T \Phi_k]_{\Omega_{k+1}^c}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega')) \quad (261)$$

Now make the change of variables

$$\Phi_{k, \Omega_{k+1}} = \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega') + Z \quad (262)$$

With the spectator fields present this says that

$$\begin{aligned} \hat{\Phi}_{k, \Omega'} &= \hat{\Psi}_{k, \Omega'}^{\text{loc}} + (0, Z) = \hat{\Psi}_{k, \Omega'} + (0, Z) + (0, \delta\Psi_{k, \Omega_{k+1}}(\Omega')) \\ \delta\Psi_{k, \Omega_{k+1}}(\Omega') &\equiv \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega') - \Psi_{k, \Omega_{k+1}}(\Omega') \end{aligned} \quad (263)$$

Then we have as a modification of (253)

$$\begin{aligned} \phi_{k, \Omega(\Lambda_k^*)} &= \phi_{k+1, \Omega'}^0 + \mathcal{Z}_{k, \Omega(\Lambda_k^*)} + \delta\phi_{k, \Omega'} \\ \delta\phi_{k, \Omega'} &= \phi_{k, \Omega(\Lambda_k^*)}(0, \delta\Psi_{k, \Omega_{k+1}}(\Omega')) \end{aligned} \quad (264)$$

We will see eventually that $\delta\Psi_{k, \Omega_{k+1}}(\Omega')$ and $\delta\phi_{k, \Omega'}$ are tiny.

Now we make the substitutions (262), (263), (264) in (240). and integrate over Z instead of $\Phi_{k, \Omega_{k+1}}$. Instead of and taking advantage of (255) we have

$$\begin{aligned} &J_{\Lambda_k, \Omega_{k+1}}^*(\Phi_{k+1}, \hat{\Psi}_{k, \Omega'} + (0, Z) + (0, \delta\Psi_{k, \Omega_{k+1}}(\Omega')), \phi_{k+1, \Omega'}^0 + \mathcal{Z}_{k, \Omega(\Lambda_k^*)} + \delta\phi_{k, \Omega'}) \\ &= J_{\Lambda_k, \Omega_{k+1}}^*(\Phi_{k+1}, \hat{\Psi}_{k, \Omega'} + (0, Z), \phi_{k+1, \Omega'}^0 + \mathcal{Z}_{k, \Omega(\Lambda_k^*)}) + R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)} \\ &= S_{k+1}^{*, 0}(\Lambda_k, \Phi_{k+1, \Omega'} + \phi_{k+1, \Omega'}^0) + \frac{1}{2} \left\langle Z, \left[\Delta_{k, \Omega(\Lambda_k^*)} + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} Z \right\rangle + R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)} + R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)} \end{aligned} \quad (265)$$

Here the first equality defines $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)}$. We also have

$$E_k^+(\Lambda_k, \phi_{k+1, \Omega'}^0 + \mathcal{Z}_{k, \Omega(\Lambda_k^*)} + \delta\phi_{k, \Omega'}) = E_k^+(\Lambda_k, \phi_{k+1, \Omega'}^0 + \mathcal{Z}_{k, \Omega(\Lambda_k^*)}) + R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)} \quad (266)$$

which defines $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)}$. The characteristic functions in Ω_{k+1} now have the form

$$\begin{aligned} \chi_k(\Lambda_k) &= \prod_{\square \in \Lambda_k} \chi_k \left((\Phi_{k, \delta\Omega_k}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega') + Z) \in \mathcal{S}_k(\square) \right) \\ \chi_k^q(\Omega_{k+1}) &= \prod_{\square \subset \Omega_{k+1}} \chi_k^q \left(\square, \Phi_{k+1}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega') + Z \right) \end{aligned} \quad (267)$$

We introduce the abbreviations

$$\begin{aligned}
S_{k+1}^{*,0}(\Lambda_k) &= S_{k+1}^{*,0}(\Lambda_k, \Phi_{k+1}, \Omega^+, \phi_{k+1}^0, \Omega') \\
E_k^+(\Lambda_k) &= E_k^+(\Lambda_k, \phi_{k+1}^0, \Omega' + \mathcal{Z}_{k, \Omega(\Lambda_k^*)}) \\
R_{k, \Pi}(\Lambda_k) &= R_{k, \Pi}(\Lambda_k, \Phi_{k, \Lambda_k - \Omega_{k+1}}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega') + Z) \\
B_{k, \Pi}(\Lambda_k) &= B_{k, \Pi}(\Lambda_k, \Phi_{k, \Omega \cap \Omega_{k+1}^c}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\Omega') + Z, W_{k, \Pi})
\end{aligned} \tag{268}$$

We also write $R_{\Pi, \Omega_{k+1}}^{(0)} = R_{k, \Pi}(\Lambda_k)$, and then tiny terms are collected in

$$R_{\Pi, \Omega_{k+1}}^{(\leq 3)} = R_{\Pi, \Omega_{k+1}}^{(0)} + R_{\Pi, \Omega_{k+1}}^{(1)} + R_{\Pi, \Omega_{k+1}}^{(2)} + R_{\Pi, \Omega_{k+1}}^{(3)} \tag{269}$$

Making all these changes (240) becomes

$$\begin{aligned}
\tilde{\rho}_{k+1}(\Phi_{k+1}) &= Z_k \mathcal{N}_{aL, \mathbb{T}_{M+N-k}}^{-1} \sum_{\Pi, \Omega_{k+1}} \int d\Phi_{k+1}^0, \Omega^{+,c} dW_{k, \Pi} K_{k, \Pi} C_{k, \Pi} C_k^q(\Lambda_k, \Omega_{k+1}) \\
&\exp\left(-S_{k+1}^{*,0}(\Lambda_k)\right) \int dZ \chi_k(\Lambda_k) \chi_k^q(\Omega_{k+1}) \exp\left(-\frac{1}{2} \left\langle Z, \left[\Delta_{k, \Omega(\Lambda_k^*)} + \frac{a}{L^2} Q^T Q\right]_{\Omega_{k+1}} Z \right\rangle\right) \\
&\exp\left(E_k^+(\Lambda_k) + R_{\Pi, \Omega_{k+1}}^{(\leq 3)} + B_{k, \Pi}(\Lambda_k)\right)
\end{aligned} \tag{270}$$

3.8 fluctuation integral

In the last expression we have the fluctuation integral with the measure $\exp(-\frac{1}{2} \langle Z, C_{k, \Omega}^{-1} Z \rangle) dZ$ where

$$C_{k, \Omega'} = \left[\Delta_{k, \Omega(\Lambda_k^*)} + \frac{a}{L^2} Q^T Q\right]_{\Omega_{k+1}}^{-1} \tag{271}$$

(If $k=0$ it is $C_{0, \Omega_1} = [-\Delta + aL^{-2} Q^T Q]_{\Omega_1}^{-1}$). This is an operator on functions on the unit lattice $\Omega_{k+1}^{(k)}$. We would like to make the change of variables $Z = C_{k, \Omega'}^{1/2} W_k$ which would yield the ultra-local measure $(\det C_{k, \Omega'}^{1/2}) \exp(-\frac{1}{2} \|W_k\|^2) dW_k$. This would move the non-locality into other terms where it is easier to handle. However non-locality inherent in $C_{k, \Omega'}^{1/2} W_k$ is awkward in the characteristic functions since it occurs in a non-analytic setting. The problem is similar to the problem of the non-local minimizer and the solution is the same.

We introduce a more local approximation to $C_{k, \Omega'}^{1/2}$, defined as follows [10]. Start with the representation

$$\begin{aligned}
C_{k, \Omega'}^{1/2} &= \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} C_{k, \Omega', r} \\
C_{k, \Omega', r} &= \left[\Delta_{k, \Omega(\Lambda_k^*)} + \frac{a}{L^2} Q^T Q + r\right]_{\Omega_{k+1}}
\end{aligned} \tag{272}$$

In appendix C we establish that

$$C_{k, \Omega', r} = \left[A_{k, r} + a_k^2 A_{k, r} Q_k G_{k, \Omega', r} Q_k^T A_{k, r}\right]_{\Omega_{k+1}} \tag{273}$$

where

$$\begin{aligned}
A_{k,r} &= \frac{1}{a_k + r} (I - Q^T Q) + \frac{1}{a_k + aL^{-2} + r} Q^T Q \\
B_{k,r} &= \frac{r}{a_k + r} (I - Q^T Q) + \frac{aL^{-2} + r}{a_k + aL^{-2} + r} Q^T Q \\
G_{k,\mathbf{\Omega}',r} &= \left[-\Delta + \bar{\mu}_k + \left[Q_{k,\mathbf{\Omega}(\Lambda_k^*)}^T \mathbf{a} Q_{k,\mathbf{\Omega}(\Lambda_k^*)} \right]_{\Omega_{k+1}^c} + a_k \left[Q_k^T B_{k,r} Q_k \right]_{\Omega_{k+1}} \right]_{\Omega_1(\Lambda_k^*)}^{-1}
\end{aligned} \tag{274}$$

Now for $0 \leq r \leq \infty$ the Green's function $G_{k,\mathbf{\Omega}',r}$ has random walk expansions just as its extreme values $G_{k+1,\mathbf{\Omega}'}^0$ and $G_{k,\mathbf{\Omega}(\Lambda_k^*)}$. We choose a version based on multiscale cubes \square in $\mathbf{\Omega}'$, just as for $G_{k+1,\mathbf{\Omega}'}^0$. (For more details see lemma 3.5 to follow). Then there is a weakened Greens function $G_{k,\mathbf{\Omega}',r}(s)$ defined for $s = \{s_\square\}$. Correspondingly we define

$$\begin{aligned}
C_{k,\mathbf{\Omega}'}^{1/2}(s) &= \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} C_{k,\mathbf{\Omega}',r}(s) \\
C_{k,\mathbf{\Omega}',r}(s) &= \left[A_{k,r} + a_k^2 A_{k,r} Q_k G_{k,\mathbf{\Omega}',r}(s) Q_k^T A_{k,r} \right]_{\Omega_{k+1}}
\end{aligned} \tag{275}$$

For \square an LM cube in Ω_{k+1} we decouple the $[r_{k+1}]$ enlargement \square^* from the complement by considering

$$G_{k,\mathbf{\Omega}',r}(\square^*) \equiv G_{k,\mathbf{\Omega}',r}(s_{\square^*} = 1, s_{(\square^*)^c} = 0) \tag{276}$$

and we have the associated $C_{k,\mathbf{\Omega}',r}(\square^*), C_{k,\mathbf{\Omega}'}^{1/2}(\square^*)$.

The local approximation on $\Omega_{k+1}^{(k)}$ is

$$(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} = \sum_{\square \subset \Omega_{k+1}} 1_\square C_{k,\mathbf{\Omega}'}^{1/2}(\square^*) \tag{277}$$

The difference

$$\delta C_{k,\mathbf{\Omega}'}^{1/2} = C_{k,\mathbf{\Omega}'}^{1/2} - (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} \tag{278}$$

is tiny as we will see.

We will also see that $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}$ is invertible, and we make the change of variables $Z = (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$ where $W_k : \Omega_{k+1}^{(k)} \rightarrow \mathbb{R}$. The quadratic form $\frac{1}{2} \langle Z, C_{k,\mathbf{\Omega}'}^{-1} Z \rangle$ becomes

$$\frac{1}{2} \langle (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k, C_{k,\mathbf{\Omega}'}^{-1} (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k \rangle = \frac{1}{2} \|W_k\|_{\Omega_{k+1}}^2 - R_{\mathbf{\Pi},\Omega_{k+1}}^{(4)} \tag{279}$$

where

$$R_{\mathbf{\Pi},\Omega_{k+1}}^{(4)} = \langle C_{k,\mathbf{\Omega}'}^{-1/2} W_k, \delta C_{k,\mathbf{\Omega}'}^{1/2} W_k \rangle - \frac{1}{2} \langle \delta C_{k,\mathbf{\Omega}'}^{1/2} W_k, C_{k,\mathbf{\Omega}'}^{-1} \delta C_{k,\mathbf{\Omega}'}^{1/2} W_k \rangle \tag{280}$$

is tiny. The change of variables also introduces

$$\det \left((C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} \right) = \det C_{k,\mathbf{\Omega}'}^{1/2} \exp(R_{\mathbf{\Pi},\Omega_{k+1}}^{(5)}) \tag{281}$$

where

$$R_{\mathbf{\Pi},\Omega_{k+1}}^{(5)} = \text{tr} \log((C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}) - \text{tr} \log(C_{k,\mathbf{\Omega}'}^{1/2}) = - \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left((C_{k,\mathbf{\Omega}'}^{-1/2} \delta C_{k,\mathbf{\Omega}'}^{1/2})^n \right) \tag{282}$$

For the last identity see (2.41) in [12]. This is also tiny.

We also want to replace $\det(C_{k,\mathbf{\Omega}'}^{1/2})$ by the global determinant $\det(C_k^{1/2})$. We have

$$\begin{aligned} & \det(C_{k,\mathbf{\Omega}'}^{1/2}) \\ &= \det(C_k^{1/2}) \exp \left(-\frac{1}{2} \operatorname{tr} \log \left(\left[\Delta_k, \mathbf{\Omega}(\Lambda_k^*) + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} \right) + \frac{1}{2} \operatorname{tr} \log \left(\Delta_k + \frac{a}{L^2} Q^T Q \right) \right) \end{aligned} \quad (283)$$

A variation of our formula for $C_{k,\mathbf{\Omega}'}^{1/2}$ is

$$\begin{aligned} & \log \left(\left[\Delta_k, \mathbf{\Omega}(\Lambda_k^*) + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} \right) \\ &= \log a_k [I - Q^T Q]_{\Omega_{k+1}} + \log(a_k + aL^{-2}) [Q^T Q]_{\Omega_{k+1}} - a_k^2 \int_0^\infty [A_{k,r} Q_k G_{k,\mathbf{\Omega}',r} Q_k^T A_{k,r}]_{\Omega_{k+1}} dr \end{aligned} \quad (284)$$

provided the integral converges. See (3.22) in [10] for the derivation. A similar formula holds for $\log \left(\Delta_k + \frac{a}{L^2} Q^T Q \right)$. Thus the exponent in (283) can be written

$$\begin{aligned} & \frac{1}{2} \log a_k \operatorname{tr} [I - Q^T Q]_{\Omega_{k+1}^c} + \frac{1}{2} \log(a_k + aL^{-2}) \operatorname{tr} [Q^T Q]_{\Omega_{k+1}^c} \\ & - \frac{1}{2} a_k^2 \int_0^\infty \operatorname{tr} [A_{k,r} Q_k G_{k,\mathbf{\Omega}',r} Q_k^T A_{k,r}]_{\Omega_{k+1}} dr + \frac{1}{2} a_k^2 \int_0^\infty \operatorname{tr} [A_{k,r} Q_k G_{k,r} Q_k^T A_{k,r}] dr \end{aligned} \quad (285)$$

But for $\Omega \subset \mathbb{T}_{M+N-k}^{-k}$, $\operatorname{tr} [Q^T Q]_\Omega$ means ⁷

$$\operatorname{tr} [Q^T Q]_{\Omega^{(k)}} = \operatorname{tr} [Q Q^T]_{\Omega^{(k+1)}} = \operatorname{tr} [I]_{\Omega^{(k+1)}} = |\Omega^{(k+1)}| = L^{-3} |\Omega^{(k)}| \quad (286)$$

and similarly $\operatorname{tr} [I - Q^T Q]_\Omega = (1 - L^{-3}) |\Omega^{(k)}|$. So the first two terms in (285) are

$$\frac{1}{2} \left((1 - L^{-3}) \log a_k + L^{-3} \log(a_k + aL^{-2}) \right) |\Omega_{k+1}^{c,(k)}| \equiv \frac{1}{2} b_k |\Omega_{k+1}^{c,(k)}| \quad (287)$$

The second two terms in (285) can be written

$$-\frac{1}{2} a_k^2 \int_0^\infty \left(\operatorname{tr} [A_{k,r} Q_k (G_{k,\mathbf{\Omega}',r} - G_{k,r}) Q_k^T A_{k,r}]_{\Omega_{k+1}} - \operatorname{tr} [A_{k,r} Q_k G_{k,r} Q_k^T A_{k,r}]_{\Omega_{k+1}^c} \right) dr \quad (288)$$

The first term here is defined to be $R_{\mathbf{\Pi},\Omega_{k+1}}^{(6)}$ and the second term is $\frac{1}{2} a_k^2 b'_k |\Omega_{k+1}^{c,(k)}|$ where

$$b'_k = \int \left(A_{k,r} Q_k G_{k,r} Q_k^T A_{k,r} \right) (x, x) dr \quad (289)$$

is independent of x . We will see that b'_k is bounded in k . Therefore we have

$$\det(C_{k,\mathbf{\Omega}'}^{1/2}) = \det(C_k^{1/2}) \exp \left(\frac{1}{2} b''_k |\Omega_k^{c,(k)}| + R_{\mathbf{\Pi},\Omega_{k+1}}^{(6)} \right) \quad (290)$$

where $b''_k = b_k + a_k^2 b'_k$.

We also introduce the Gaussian measure with identity covariance

$$d\mu_{\Omega_{k+1}}(W_k) = (2\pi)^{-\frac{1}{2} |\Omega_{k+1}^{(k)}|} \exp \left(-\frac{1}{2} \|W_k\|^2 \right) dW_k \quad (291)$$

⁷ Keep in mind that for $\Omega \subset \mathbb{T}_{M+N-k}^{-k}$ we have $|\Omega^{(k)}| = \operatorname{Vol}(\Omega)$. We prefer to write it in the first form which is scale invariant.

Normalization factors are rearranged as

$$(2\pi)^{\frac{1}{2}|\Omega_{k+1}^{(k)}|} = (2\pi)^{\frac{1}{2}|\mathbb{T}_{M+N-k}^0|} (2\pi)^{-\frac{1}{2}|\Omega_{k+1}^{c,(k)}|} \quad (292)$$

The first term contributes to

$$Z_{k+1}^0 \equiv \mathcal{N}_{aL, \mathbb{T}_{M+N-k}^1}^{-1} (2\pi)^{\frac{1}{2}|\mathbb{T}_{M+N-k}^0|} (\det C_k)^{1/2} Z_k \quad (293)$$

The second term combines with $\exp\left(\frac{1}{2}b_k''|\Omega_k^{c,(k)}|\right)$ to give $\exp\left(\frac{1}{2}c_{k+1}|\Omega_k^{c,(k)}|\right)$ where $c_{k+1} = b_k'' - \log 2\pi$ is bounded in k .

There are still more changes. The field $\mathcal{Z}_{k,\Omega(\Lambda_k^*)} = a_k G_{k,\Omega(\Lambda_k^*)} Q_k^T Z$ now becomes

$$\begin{aligned} \mathcal{W}_{k,\Omega'}^{\text{loc}} &\equiv a_k G_{k,\Omega(\Lambda_k^*)} Q_k^T (C_{k,\Omega'}^{1/2})^{\text{loc}} W_k \\ &= a_k G_{k,\Omega(\Lambda_k^*)} Q_k^T C_{k,\Omega'}^{1/2} W_k + a_k G_{k,\Omega(\Lambda_k^*)} Q_k^T \delta C_{k,\Omega'}^{1/2} W_k \\ &\equiv \mathcal{W}_{k,\Omega'} + \delta \mathcal{W}_{k,\Omega'} \end{aligned} \quad (294)$$

and $\delta \mathcal{W}_{k,\Omega'}$ will be tiny. Therefore we can write

$$E_k^+(\Lambda_k, \phi_{k+1}^0, \Omega' + \mathcal{W}_{k,\Omega'}^{\text{loc}}) = E_k^+(\Lambda_k, \phi_{k+1}^0, \Omega' + \mathcal{W}_{k,\Omega'}) + R_{\Pi, \Omega_{k+1}}^{(7)} \quad (295)$$

which defines a tiny term $R_{\Pi, \Omega_{k+1}}^{(7)}$.

Now we rewrite (270). The characteristic functions $\chi_k(\Omega_{k+1}), \chi_k^q(\Omega_{k+1})$ are the same as (268), except that Z is replaced by $(C_{k,\Omega'}^{1/2})^{\text{loc}} W_k$. Now $E_k^+(\Lambda_k)$ stands for $E_k^+(\Lambda_k, \phi_{k+1}^0, \Omega' + \mathcal{W}_{k,\Omega'})$. Also $R_{\Pi, \Omega_{k+1}}^{(\leq 3)}, B_{k,\Pi}(\Lambda_k)$ are the same as before, but with Z replaced by $(C_{k,\Omega'}^{1/2})^{\text{loc}} W_k$ and $\mathcal{Z}_{k,\Omega(\Lambda_k^*)}$ replaced by $\mathcal{W}_{k,\Omega'}^{\text{loc}}$. All tiny terms are collected in $R_{\Pi, \Omega_{k+1}}^{(\leq 7)} = R_{\Pi, \Omega_{k+1}}^{(0)} + \dots + R_{\Pi, \Omega_{k+1}}^{(7)}$. (Actually parts of $R_{\Pi, \Omega_{k+1}}^{(6)}$ are not tiny, but these will eventually end up as boundary terms.) Making all these changes (270) becomes

$$\begin{aligned} \tilde{\rho}_{k+1}(\Phi_{k+1}) &= Z_{k+1}^0 \sum_{\Pi, \Omega_{k+1}} \int d\Phi_{k+1, \Omega_{k+1}}^0 dW_{k,\Pi} K_{k,\Pi} \mathcal{C}_{k,\Pi} \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) \\ &\quad \exp\left(-S_{k+1}^{*,0}(\Lambda_k)\right) \exp\left(c_{k+1}|\Omega_{k+1}^{c,(k)}|\right) \\ &\quad \int d\mu_{\Omega_{k+1}}(W_k) \chi_k(\Lambda_k) \chi_k^q(\Omega_{k+1}) \exp\left(E_k^+(\Lambda_k) + R_{\Pi, \Omega_{k+1}}^{(\leq 7)} + B_{k,\Pi}(\Lambda_k)\right) \end{aligned} \quad (296)$$

3.9 estimates

We collect some estimates on these operators. First we elaborate on the statement that $G_{k,\Omega',r}$ has a random walk expansion. This means repeating the analysis of section 2.5 with some modifications. Actually things are a little easier here since we will not need derivatives and L^2 bounds will suffice. We will be a little more general and consider $G_{k,\Omega^+,r}$ where $\Omega^+ = (\Omega, \Omega_{k+1}) = (\Omega_1, \dots, \Omega_{k+1})$ satisfies the minimal separation conditions (96) and where

$$G_{k,\Omega^+,r} = \left[-\Delta + \bar{\mu}_k + \left[Q_{k,\Omega}^T \mathbf{a} Q_{k,\Omega} \right]_{\Omega_{k+1}^c} + a_k \left[Q_k^T B_{k,r} Q_k \right]_{\Omega_{k+1}} \right]_{\Omega_1}^{-1} \quad (297)$$

The inverse taken with Dirichlet boundary conditions. Instead of theorem 2.2 we have:

Lemma 3.5. *The Green's function $G_{k,\mathbf{\Omega}^+,r}$ has a random walk expansion*

$$G_{k,\mathbf{\Omega}^+,r} = \sum_{\omega} G_{k,\mathbf{\Omega}^+,r,\omega} \quad (298)$$

based on multiscale cubes \square for $\mathbf{\Omega}^+$. It converges in L^2 norm for M sufficiently large and yields the following bounds. There are constants C (depending on L) and $\gamma = \mathcal{O}(L^{-2})$ so for $\Delta_y \subset \delta\Omega_j$ and $\Delta_{y'} \subset \delta\Omega_{j'}$ and all $r \geq 0$

$$\|1_{\Delta_y} G_{k,\mathbf{\Omega}^+,r} 1_{\Delta_{y'}} f\|_2 \leq CL^{-2(k-j')} e^{-\gamma d_{\mathbf{\Omega}^+}(y,y')} \|f\|_2 \quad (299)$$

Proof. The proof follows the analysis in section 2.5, see also [10]. First note that

$$a_k Q_k^T B_{k,r} Q_k = \frac{a_k r}{a_k + r} Q_k^T Q_k + \frac{a_k^2 a L^{-2}}{(a_k + r)(a_k + a L^{-2} + r)} Q_{k+1}^T Q_{k+1} \quad (300)$$

This is bounded below by $\mathcal{O}(1)L^{-2}Q_{k+1}^T Q_{k+1}$ for $0 \leq r \leq 1$ and by $\mathcal{O}(1)Q_k^T Q_k$ for $r \geq 1$. It follows that for an LM cube \square with $\tilde{\square} \subset \Omega_{k+1}$, and Neumann boundary conditions on $\tilde{\square}$,

$$\left[-\Delta + \bar{\mu}_k + \left[Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}} \right]_{\Omega_{k+1}^c} + a_k \left[Q_k^T B_{k,r} Q_k \right]_{\Omega_{k+1}} \right]_{\tilde{\square}} \geq \mathcal{O}(1)[-\Delta + \mathcal{O}(1)L^{-2}]_{\tilde{\square}} \quad (301)$$

If $G_{k,\mathbf{\Omega}^+,r}(\tilde{\square})$ is the the inverse of the operator on the left, then

$$\|G_{k,\mathbf{\Omega}^+,r}(\tilde{\square})f\|_2, \|\partial G_{k,\mathbf{\Omega}^+,r}(\tilde{\square})f\|_2 \leq C\|f\|_2 \quad (302)$$

This can be extended to all $\square \subset \Omega_{k+1}$. More generally for an $L^{-(k-j)}M$ cube in $\square \subset \delta\Omega_j$ we have

$$\|G_{k,\mathbf{\Omega}^+,r}(\tilde{\square})f\|_2, L^{-(k-j)}\|\partial G_{k,\mathbf{\Omega}^+,r}(\tilde{\square})f\|_2 \leq CL^{-2(k-j)}\|f\|_2 \quad (303)$$

This improves to the local bound for scaled cubes $\Delta_y \subset \delta\Omega_j$ and $\Delta_{y'} \subset \delta\Omega_{j'}$ and $|j - j'| \leq 1$

$$\begin{aligned} \|1_{\Delta_y} G_{k,\mathbf{\Omega}^+,r}(\tilde{\square}) 1_{\Delta_{y'}} f\|_2 &\leq CL^{-2(k-j')} e^{-\gamma d_{\mathbf{\Omega}^+}(y,y')} \|f\|_2 \\ L^{-(k-j)} \|1_{\Delta_y} \partial G_{k,\mathbf{\Omega}^+,r}(\tilde{\square}) 1_{\Delta_{y'}} f\|_2 &\leq CL^{-2(k-j')} e^{-\gamma d_{\mathbf{\Omega}^+}(y,y')} \|f\|_2 \end{aligned} \quad (304)$$

To see this for $\square \subset \Omega_{k+1}$ one shows that the bound (301) still holds when the left side is replaced by $e^{-q \cdot x}[\dots]e^{q \cdot x}$ for $q = \mathcal{O}(L^{-2})$. This yields a bound on $\|1_{\Delta_y} G_{k,\mathbf{\Omega}^+,r}(\tilde{\square}) 1_{\Delta_{y'}} f\|_2$ with a factor $e^{q \cdot (y-y')}$ and one chooses q in the direction $-(y - y')$. See the appendix E of part I for more details on this type of argument.

The random walk expansion is based on the estimates (304) as in theorem 2.2. Now estimates are all in L^2 norms and the bound (299) follows. This completes the proof.

Lemma 3.6.

1. For $f : \Omega_{k+1}^{(k)} \rightarrow \mathbb{R}$

$$\begin{aligned} \left| C_{k,\mathbf{\Omega}'}^{1/2} f \right|, \left| (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} f \right| &\leq C\|f\|_{\infty} \\ |\delta C_{k,\mathbf{\Omega}'}^{1/2} f| &\leq C e^{-r_{k+1}} \|f\|_{\infty} \end{aligned} \quad (305)$$

2. $C_{k,\mathbf{\Omega}'}^{1/2}$ and $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}$ are invertible and

$$\left| C_{k,\mathbf{\Omega}'}^{-1/2} f \right|, \left| \left[(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} \right]^{-1} f \right| \leq C \|f\|_{\infty} \quad (306)$$

Proof. First define $D_{k,\mathbf{\Omega}',r} = [Q_k G_{k,\mathbf{\Omega}',r} Q_k^T]_{\Omega_{k+1}}$. This has the kernel for $y, y' \in \Omega_{k+1}^{(k)}$

$$D_{k,\mathbf{\Omega}',r}(y, y') = \langle Q_k^T \delta_y, G_{k,\mathbf{\Omega}',r} Q_k^T \delta_{y'} \rangle = \langle 1_{\Delta_y}, G_{k,\mathbf{\Omega}',r} 1_{\Delta_{y'}} \rangle \quad (307)$$

where $\Delta_y, \Delta_{y'}$ are unit cubes. By the lemma with $j = j' = k+1$ this satisfies

$$|D_{k,\mathbf{\Omega}',r}(y, y')| \leq C \|1_{\Delta_y}\|_2 \|1_{\Delta_{y'}}\|_2 e^{-\gamma d_{\mathbf{\Omega}'}(y, y')} = C e^{-\gamma d(y, y')} \quad (308)$$

and so $|D_{k,\mathbf{\Omega}',r} f| \leq C \|f\|_{\infty}$. Combining this with $|A_{k,r} f| \leq \mathcal{O}(1)(1+r)^{-1} \|f\|_{\infty}$ we can estimate $C_{k,\mathbf{\Omega}',r} = A_{k,r} + a_k^2 A_{k,r} D_{k,\mathbf{\Omega}',r} A_{k,r}$. by $|C_{k,\mathbf{\Omega}',r} f| \leq C(1+r)^{-1} \|f\|_{\infty}$. This gives $|C_{k,\mathbf{\Omega}',r}^{1/2} f| \leq C \|f\|_{\infty}$.

The same estimates hold for the weakened versions based on $G_{k,\mathbf{\Omega}',r}(s)$. These are denoted $D_{k,\mathbf{\Omega}',r}(s)$, $C_{k,\mathbf{\Omega}',r}(s)$, $C_{k,\mathbf{\Omega}',r}^{1/2}(s)$ and satisfy the same bounds. Specializing to $s_{\square^*} = 1, s_{(\square^*)^c} = 0$ we get the same bounds for $G_{k,\mathbf{\Omega}',r}(\square^*)$ and hence for $D_{k,\mathbf{\Omega}',r}(\square^*)$, $C_{k,\mathbf{\Omega}',r}(\square^*)$ and $C_{k,\mathbf{\Omega}',r}^{1/2}(\square^*)$. The bound on $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}$ follows from the bound on $C_{k,\mathbf{\Omega}'}^{1/2}(\square^*)$.

The bound on $\delta C_{k,\mathbf{\Omega}'}^{1/2} = C_{k,\mathbf{\Omega}'}^{1/2} - (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}$ follows from a modification of lemma 3.5 which says for $y, y' \in \square \subset \Omega_{k+1}$

$$\|1_{\Delta_y} (G_{k,\mathbf{\Omega}',r}(\square^*) - G_{k,\mathbf{\Omega}',r}) 1_{\Delta_{y'}} f\|_2 \leq C e^{-r_{k+1}} e^{-\gamma d_{\mathbf{\Omega}'}(y, y')} \|f\|_2 \quad (309)$$

This is true since in the random walk expansion for the difference, any path must start in \square , exit \square^* , and then return to \square . Thus the minimum number of steps is at approximately $2[r_{k+1}]$ and this enables us to extract a factor $e^{-r_{k+1}}$. Running through the above argument gives a bound $|(C_{k,\mathbf{\Omega}'}^{1/2} - C_{k,\mathbf{\Omega}'}^{1/2}(\square^*)) f| \leq C e^{-r_{k+1}} \|f\|_{\infty}$ which gives the bound on $\delta C_{k,\mathbf{\Omega}'}^{1/2}$. This completes the proof of part 1.

For part 2, $C_{k,\mathbf{\Omega}'}^{1/2}$ is invertible since $C_{k,\mathbf{\Omega}'}$ is invertible and we can write

$$C_{k,\mathbf{\Omega}'}^{-1/2} = C_{k,\mathbf{\Omega}'}^{-1} C_{k,\mathbf{\Omega}'}^{1/2} = \left[\Delta_{k,\mathbf{\Omega}(\Lambda_k^*)} + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} C_{k,\mathbf{\Omega}'}^{1/2} \quad (310)$$

Restricted to Ω_{k+1} we have $\Delta_{k,\mathbf{\Omega}(\Lambda_k^*)} = a_k + a_k^2 Q_k G_{k,\mathbf{\Omega}(\Lambda_k^*)} Q_k^T$. Since $|G_{k,\mathbf{\Omega}(\Lambda_k^*)} f| \leq C \|f\|_{\infty}$ follows from theorem 2.2 we have

$$\left| \left[\Delta_{k,\mathbf{\Omega}} + \frac{a}{L^2} Q^T Q \right]_{\Omega_{k+1}} f \right| \leq C \|f\|_{\infty} \quad (311)$$

Combined with the bound on $C_{k,\mathbf{\Omega}'}^{1/2}$, this yields $|C_{k,\mathbf{\Omega}'}^{-1/2} f| \leq C \|f\|_{\infty}$

Finally we write $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} = C_{k,\mathbf{\Omega}'}^{1/2} - \delta C_{k,\mathbf{\Omega}'}^{1/2}$ and then the inverse is realized as

$$\left[(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} \right]^{-1} = C_{k,\mathbf{\Omega}'}^{-1/2} \sum_{n=0}^{\infty} (\delta C_{k,\mathbf{\Omega}'}^{1/2} C_{k,\mathbf{\Omega}'}^{-1/2})^n \quad (312)$$

The convergence and the bound then follow from $|C_{k,\mathbf{\Omega}'}^{-1/2} f| \leq C \|f\|_{\infty}$ and $|\delta C_{k,\mathbf{\Omega}'}^{1/2} f| \leq e^{-r_{k+1}} \|f\|_{\infty}$, since we can assume $C e^{-r_{k+1}} < \frac{1}{2}$.

Lemma 3.7. $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(5)}$ has a local expansion in LM cubes

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(5)} = \sum_{\square \subset \Omega_{k+1}} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(5)}(\square) \quad |R_{\mathbf{\Pi}, \Omega_{k+1}}^{(5)}(\square)| \leq C(LM)^3 e^{-\frac{1}{2}r_{k+1}} \quad (313)$$

Proof. $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(5)}$ given by (282) has the local expansion

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(5)}(\square) = - \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} \left(1_{\square} (C_{k, \mathbf{\Omega}'}^{-1/2} \delta C_{k, \mathbf{\Omega}'}^{1/2})^n \right) = - \sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \square} \left((C_{k, \mathbf{\Omega}'}^{-1/2} \delta C_{k, \mathbf{\Omega}'}^{1/2})^n \delta_y \right)(y) \quad (314)$$

From the bounds (305) and (306) we have

$$\left| \left((C_{k, \mathbf{\Omega}'}^{-1/2} \delta C_{k, \mathbf{\Omega}'}^{1/2})^n \delta_y \right)(y) \right| \leq (C e^{-r_{k+1}})^n \|\delta_y\|_{\infty} \leq (C e^{-r_{k+1}})^n \quad (315)$$

Summing over $y \in \square$ gives the factor $(LM)^3$ and summing over n gives the result.

Lemma 3.8. $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}$ has a local expansion in LM cubes

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)} = \sum_{\square \subset \Omega_{k+1}} R_{k, \mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square) \quad |R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square)| \leq C(LM)^3 \quad (316)$$

If $\square \subset \Omega_{k+1} - \Omega_{k+1}^{\natural}$ then the bound improves to

$$|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square)| \leq C(LM)^3 e^{-r_{k+1}} \quad (317)$$

Proof. $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}$ is given in (288). With $D_{k, \mathbf{\Omega}', r} = [Q_k G_{k, \mathbf{\Omega}', r} Q_k^T]_{\Omega_{k+1}}$ and $D_{k, r} = [Q_k G_{k, r} Q_k^T]_{\Omega_{k+1}}$ the expansion holds with

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square) = -\frac{1}{2} a_k^2 \int_0^{\infty} \sum_{y \in \square} \left(A_{k, r} (D_{k, \mathbf{\Omega}', r} - D_{k, r}) A_{k, r} \right)(y, y) dr \quad (318)$$

Since $\|A_{k, r} \delta_y\|_2 \leq \mathcal{O}(1)(1+r)^{-1}$ and $|D_{k, \mathbf{\Omega}', r}(y, y') - D_{k, r}(y, y')| \leq C e^{-\gamma d(y, y')}$ we have

$$\left| \left\langle A_{k, r} \delta_y, \left(D_{k, \mathbf{\Omega}', r} - D_{k, r} \right) A_{k, r} \delta_y \right\rangle \right| \leq C \|A_{k, r} \delta_y\|_2^2 \leq C \frac{1}{(1+r)^2} \quad (319)$$

This gives the announced

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square) \leq C(LM)^3 \int_0^{\infty} \frac{1}{(1+r)^2} \leq C(LM)^3 \quad (320)$$

If $\square \subset \Omega_{k+1}^{\natural}$ then for $y, y' \in \square$

$$\|1_{\Delta_y} (G_{k, \mathbf{\Omega}', r} - G_{k, r}) 1_{\Delta_{y'}} f\|_2 \leq C e^{-r_{k+1}} e^{-\gamma d(y, y')} \|f\|_2 \quad (321)$$

This holds since in the random walk expansion for $1_{\Delta_y} (G_{k, \mathbf{\Omega}', r} - G_{k, r}) 1_{\Delta_{y'}}$ any path must start in Ω_{k+1}^{\natural} , pass through Ω_{k+1}^c , and then return to Ω_{k+1}^{\natural} . But Ω_{k+1}^{\natural} and Ω_{k+1}^c are separated by $[r_{k+1}]$ layers of LM cubes. Thus the minimum number of steps is approximately $2[r_{k+1}]$, which allows us to extract a factor $e^{-r_{k+1}}$. This gives an extra factor of $e^{-r_{k+1}}$ in the estimate on $|D_{k, \mathbf{\Omega}', r}(y, y') - D_{k, r}(y, y')|$ and hence in the result.

Similar estimates show that b'_k is bounded.

3.10 new characteristic functions

In Ω_{k+1} the characteristic functions are limitations on the field $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') + (C_{k,\Omega'}^{1/2})^{\text{loc}} W_k$ and hence on the background field $\phi_{k+1,\Omega'}^0(\square^*)$ and the fluctuation field W_k . This structure is awkward and we replace it by separate cleaner characteristic functions for the background and fluctuation.

Anticipating that the term $\exp(-S_{k+1}^{*,0}(\Lambda_k))$ in (296) suppresses large fields, we introduce for each LM cube \square well inside Ω_{k+1} , the characteristic function $\chi_{k+1}^0(\square)$ enforcing the inequalities in $\tilde{\square}$:

$$\begin{aligned} |\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega^+(\square)}^0| &\leq p_{k+1}L^{-\frac{1}{2}} \\ |\partial\phi_{k+1,\Omega^+(\square)}^0| &\leq p_{k+1}L^{-\frac{3}{2}} \\ |\phi_{k+1,\Omega^+(\square)}^0| &\leq p_{k+1}\lambda_{k+1}^{-1/4}L^{-\frac{1}{2}} \end{aligned} \quad (322)$$

Here $\Omega^+(\square)$ has $k+1$ layers and $\phi_{k+1,\Omega^+(\square)}^0 = \phi_{k+1,\Omega^+(\square)}^0(\tilde{Q}_{\mathbb{T}^1,\Omega^+(\square)}^T \Phi_{k+1})$ as defined in (55). Call functions satisfying (322) $\mathcal{S}_{k+1}^0(\square)$, since when we scale later on these will become the inequalities defining $\mathcal{S}_{k+1}(\square)$, and $\chi_{k+1}^0(\square)$ will scale to $\chi_{k+1}(\square)$.

Now we write with $\zeta_{k+1}^0(\square) = 1 - \chi_{k+1}^0(\square)$

$$\begin{aligned} 1 &= \prod_{\square \subset \Omega_{k+1}^{\natural}} \zeta_{k+1}^0(\square) + \chi_{k+1}^0(\square) \\ &= \sum_{Q_{k+1} \subset \Omega_{k+1}^{\natural}} \prod_{\square \subset Q_{k+1}} \zeta_{k+1}^0(\square) \prod_{\square \subset \Omega_{k+1}^{\natural} - Q_{k+1}} \chi_{k+1}^0(\square) \\ &\equiv \sum_{Q_{k+1} \subset \Omega_{k+1}^{\natural}} \zeta_{k+1}^0(Q_{k+1}) \chi_{k+1}^0(\Omega_{k+1}^{\natural} - Q_{k+1}) \end{aligned} \quad (323)$$

Anticipating that the factor $\exp(-1/2\|W_k\|_{\Omega_{k+1}}^2)$ in (296) suppresses large W_k we define

$$\chi_k^w(\square, W_k) = \prod_{x \in \square} \chi(|W_k(x)| \leq p_{0,k}) \quad (324)$$

for $p_{0,k} \leq p_k$. Then we write with $\zeta_k^w(\square) = 1 - \chi_k^w(\square)$

$$\begin{aligned} 1 &= \prod_{\square \subset \Omega_{k+1}} \zeta_k^w(\square) + \chi_k^w(\square) \\ &= \sum_{R_{k+1} \subset \Omega_{k+1}} \prod_{\square \subset R_{k+1}} \zeta_k^w(\square) \prod_{\square \subset \Omega_{k+1} - R_{k+1}} \chi_k^w(\square) \\ &\equiv \sum_{R_{k+1} \subset \Omega_{k+1}} \zeta_k^w(R_{k+1}) \chi_k^w(\Omega_{k+1} - R_{k+1}) \end{aligned} \quad (325)$$

Now we have

$$1 = \sum_{Q_{k+1}, R_{k+1}} \zeta_{k+1}^0(Q_{k+1}) \zeta_k^w(R_{k+1}) \chi_{k+1}^0(\Omega_{k+1}^{\natural} - Q_{k+1}) \chi_k^w(\Omega_{k+1} - R_{k+1}) \quad (326)$$

The new large field regions Q_{k+1}, R_{k+1} generate a new small field region Λ_{k+1} , also a union of LM cubes, defined by

$$\Lambda_{k+1} = \Omega_{k+1}^{5\natural} - (Q_{k+1}^{5*} \cup R_{k+1}^{5*}) \quad \text{or} \quad \Lambda_{k+1}^c = (\Omega_{k+1}^c)^{5*} \cup Q_{k+1}^{5*} \cup R_{k+1}^{5*} \quad (327)$$

We write $Q_{k+1}, R_{k+1} \rightarrow \Lambda_{k+1}$ and classify the terms in (326) by the Λ_{k+1} that they generate. Thus we have

$$1 = \sum_{\Lambda_{k+1} \subset \Omega_{k+1}^{5\sharp}} \mathcal{C}_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) \quad (328)$$

where

$$\mathcal{C}_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) = \sum_{Q_{k+1}, R_{k+1} \rightarrow \Lambda_{k+1}} \zeta_{k+1}^0(Q_{k+1}) \zeta_k^w(R_{k+1}) \chi_{k+1}^0(\Omega_{k+1}^\sharp - Q_{k+1}) \chi_k^w(\Omega_{k+1} - R_{k+1}) \quad (329)$$

The sum is still restricted by $P_{k+1} \subset \Omega_{k+1}^\sharp$ and $Q_{k+1} \subset \Omega_{k+1}$.

Note that $\mathcal{C}_{k+1}(\Omega_{k+1}, \Lambda_{k+1})$ enforces that the bounds (322) and $|W_k| \leq p_{0,k}$ hold on Λ_{k+1}^{4*} . To see this it suffices to show that every term in the sum (329) has this property. But the term with Q_{k+1}, R_{k+1} enforces the bounds (322) on $\Omega_{k+1}^\sharp - Q_{k+1}$ and the bound $|W_k| \leq p_{0,k}$ on $\Omega_{k+1} - R_{k+1}$. Since both these sets contain Λ_{k+1}^{4*} we have the result.

We insert (328) under the integral sign in (296). The sum is now over

$$\Pi^+ = (\Pi, \Omega_{k+1}, \Lambda_{k+1}) = (\Omega_1, \Lambda_1, \dots, \Omega_{k+1}, \Lambda_{k+1}) \quad (330)$$

and

$$\begin{aligned} \tilde{\rho}_{k+1}(\Phi_{k+1}) = & Z_{k+1}^0 \sum_{\Pi^+} \int d\Phi_{k+1, \Omega_{k+1}^{+,c}}^0 dW_{k, \Pi} K_{k, \Pi} \mathcal{C}_{k, \Pi} \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) \mathcal{C}_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) \\ & \exp\left(-S_{k+1}^{*,0}(\Lambda_k)\right) \exp\left(c_{k+1}|\Omega_{k+1}^{c,(k)}|\right) \\ & \int d\mu_{\Omega_{k+1}}(W_k) \chi_k(\Lambda_k) \chi_k^q(\Omega_{k+1}) \exp\left(E_k^+(\Lambda_k) + R_{\Pi, \Omega_{k+1}}^{(\leq 7)} + B_{k, \Pi}(\Lambda_k)\right) \end{aligned} \quad (331)$$

3.11 more estimates

We collect some estimates for future use.

3.11.1 bounds on the fluctuation field

The characteristic functions can now be written

$$\left[\mathcal{C}_{k, \Pi} \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) \chi_k(\Lambda_k) \chi_k^q(\Omega_{k+1}) \right] \mathcal{C}_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) \quad (332)$$

The bracketed characteristic functions still enforce the bounds of lemma 3.4, except that the bound (242) on Φ_k now only holds on $\tilde{\Lambda}_k - \Omega_{k+1}$ since Φ_k on Ω_{k+1} was relabeled. The relabeling does not affect any of the remaining bounds. In a slightly weaker form we can summarize them as

$$|\Phi_k| \leq Cp_k \lambda_k^{-1/4} \quad |\partial \Phi_k| \leq Cp_k \quad \text{on } \Omega_k - \Omega_{k+1} \quad (333)$$

$$|\Phi_{k+1}| \leq 3p_k \lambda_k^{-1/4} \quad |\partial \Phi_{k+1}| \leq 4p_k \quad \text{on } \Omega_{k+1} \quad (334)$$

as well as

$$|\Phi_{k+1}^\#| \leq Cp_k \lambda_k^{-1/4} \quad |\partial \Phi_{k+1}^\#| \leq Cp_k \quad \text{on } \Omega_k \quad (335)$$

Our immediate goal is to get a bound on W_k on its whole domain Ω_{k+1} . We start with:

Lemma 3.9. *The bracketed characteristic functions in (332) enforce the following inequalities:*

1. For an LM-cube $\square \subset \Omega_{k+1}$

$$\begin{aligned} |\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega'}^0| &\leq Cp_k & \text{on } \tilde{\square} \cap \Omega_{k+1} \\ |\partial\phi_{k+1,\Omega'}^0| &\leq Cp_k & \text{on } \tilde{\square} \\ |\phi_{k+1,\Omega'}^0| &\leq Cp_k\lambda_k^{-\frac{1}{4}} & \text{on } \tilde{\square} \end{aligned} \quad (336)$$

The same bounds hold for $\phi_{k,\Omega'}^0(s), \phi_{k,\Omega'}^0(\square^*)$ and

$$\left| \phi_{k+1,\Omega'}^0(\square^*) - \phi_{k+1,\Omega'}^0 \right| \leq e^{-r_{k+1}} \quad (337)$$

2. For $\square \subset \Omega_{k+1}$ we have on $\tilde{\square} \cap \Omega_{k+1}$

$$|\Psi_{k,\Omega_{k+1}}(\Omega')| \leq Cp_k\lambda_k^{-1/4} \quad |\partial\Psi_{k,\Omega_{k+1}}(\Omega')| \leq Cp_k \quad (338)$$

The same holds for $\Psi_{k,\Omega_{k+1}}(\Omega', s), \Psi_{k,\Omega_{k+1}}(\Omega', \square^*)$ and $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega')$ and

$$|\delta\Psi_{k,\Omega_{k+1}}(\Omega')| = |\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') - \Psi_{k,\Omega_{k+1}}(\Omega')| \leq e^{-r_{k+1}} \quad (339)$$

Proof. Instead of considering $\phi_{k+1,\Omega'}^0(\hat{\Phi}_{k+1,\Omega'})$ with $\hat{\Phi}_{k+1,\Omega'} = (\tilde{Q}_{\mathbb{T}^0,\Omega(\Lambda_k^*)}^T \Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}})$ we start with $\phi_{k+1,\Omega'}^0(\Phi_{k+1,\Omega'}^*)$ where

$$\Phi_{k+1,\Omega'}^* = \tilde{Q}_{\mathbb{T}^1,\Omega'}^T(Q\Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}}) = \tilde{Q}_{\mathbb{T}^1,\Omega'}^T\Phi_{k+1}^\# \quad (340)$$

We claim that the bounds (336) hold for $\phi_{k+1,\Omega'}^0(\Phi_{k+1,\Omega'}^*)$. This follows from the bounds on $\Phi_{k+1}^\#$ by an argument very similar to lemma 3.2. The differences are that we are considering the pre-scaled field $\phi_{k+1,\Omega'}^0$ and we need the bounds on the basic fields on the larger set $\Omega'_1 = \Omega_1(\Lambda_k^*) \subset \Omega_k$.

Next note that since $Q_{\mathbb{T}^1,\Omega'}$ is the identity on Ω_{k+1} and since it is $QQ_{\mathbb{T}^0,\Omega(\Lambda_k^*)}$ on $\delta\Omega_k$ we have

$$\hat{\Phi}_{k+1,\Omega'} - \Phi_{k+1,\Omega'}^* = \left(\tilde{Q}_{\mathbb{T}^0,\Omega(\Lambda_k^*)}^T(I - Q^TQ)\Phi_{k,\delta\Omega_k}, 0 \right) \quad (341)$$

The expression $(I - Q^TQ)\Phi_{k,\delta\Omega_k}$ can be written as a function of $\partial\Phi_{k,\delta\Omega_k}$. Using the bound on this function it is bounded by Cp_k . Therefore we have by (142)

$$\begin{aligned} |\phi_{k+1,\Omega'}^0(\hat{\Phi}_{k+1,\Omega'}) - \phi_{k+1,\Omega'}^0(\Phi_{k+1,\Omega'}^*)| &\leq Cp_k \\ |\partial\phi_{k+1,\Omega'}^0(\hat{\Phi}_{k+1,\Omega'}) - \partial\phi_{k+1,\Omega'}^0(\Phi_{k+1,\Omega'}^*)| &\leq Cp_k \end{aligned} \quad (342)$$

Hence the bounds (336) also hold for $\phi_{k+1,\Omega'}^0(\hat{\Phi}_{k+1,\Omega'})$ as claimed.

This argument holds equally well for the weak version $\phi_{k,\Omega'}^0(s)$ and $\phi_{k,\Omega'}^0(\square^*)$ is a special case.. To bound the difference $\phi_{k+1,\Omega'}^0(\square^*) - \phi_{k+1,\Omega'}^0$ we claim that for $y, y' \in \tilde{\square}$

$$|1_{\Delta_y}(G_{k+1,\Omega'}^0 - G_{k+1,\Omega'}^0(\square^*))1_{\Delta_{y'}}f| \leq Ce^{-r_{k+1}}e^{-\gamma^d\Omega'(y,y')}\|f\|_\infty \quad (343)$$

This is a variation of (141) with the following observation. In the random walk expansion for this expression we start and end in $\tilde{\square}$, and only terms which exit \square^* contribute. These paths must

have at least $[r_{k+1}]$ steps. Hence we can extract a factor $M^{-\frac{1}{4}[r_{k+1}]} \leq e^{-2r_{k+1}}$ when estimating the expansion. With this modification the bound (337) follows as in (142); here we use $e^{-r_{k+1}} Cp_k \lambda_k^{-\frac{1}{4}} \leq 1$.

The minimizer can be written

$$\Psi_{k,\Omega_{k+1}}(\Omega') = Q_k \phi_{k+1,\Omega'}^0 + \frac{aL^{-2}}{a_k + aL^{-2}} Q^T \left(\Phi_{k+1} - Q_{k+1} \phi_{k+1,\Omega'}^0 \right) \quad (344)$$

We claim that on $\tilde{\square} \cap \Omega_{k+1}$

$$|\Psi_{k,\Omega_{k+1}}(\Omega')| \leq Cp_k \lambda_k^{-1/4} \quad |\partial \Psi_{k,\Omega_{k+1}}(\Omega')| \leq Cp_k \quad (345)$$

These follow more or less directly from the bounds (336). The bound $|Q^T(\Phi_{k+1} - Q_{k+1} \phi_{k+1,\Omega'}^0)| \leq Cp_k$, implies a bound of the same form on the derivative, since it is defined on a unit lattice. The same argument gives bounds on $\Psi_{k,\Omega_{k+1}}(\Omega', s)$ and $\Psi_{k,\Omega_{k+1}}(\Omega', \square^*)$. The bound $|\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega')| \leq Cp_k \lambda_k^{-1/4}$ follows as well.

We also have by (337) on $\tilde{\square} \cap \Omega_{k+1}$

$$\begin{aligned} & |\Psi_{k,\Omega_{k+1}}(\Omega', \square^*) - \Psi_{k,\Omega_{k+1}}(\Omega')| \\ &= \left| Q_k \left(\phi_{k+1,\Omega'}^0(\square^*) - \phi_{k+1,\Omega'}^0 \right) - \frac{aL^{-2}}{a_k + aL^{-2}} Q^T Q_{k+1} \left(\phi_{k+1,\Omega'}^0(\square^*) - \phi_{k+1,\Omega'}^0 \right) \right| \leq e^{-r_{k+1}} \end{aligned} \quad (346)$$

This implies the bound on $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') - \Psi_{k,\Omega_{k+1}}(\Omega')$

Finally we need the bound $|\partial \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega')| \leq Cp_k$. There is a potential problem here when the derivative crosses the boundary of a cube \square . Suppose \square_1, \square_2 are adjacent cubes and $x \in \square_1 \cap \Omega_{k+1}^{(k)}$ and $x + e_\mu \in \square_2 \cap \Omega_{k+1}^{(k)}$. We need to show $|\Psi_{k,\Omega_{k+1}}(\Omega', \square_2^*, x + e_\mu) - \Psi_{k,\Omega_{k+1}}(\Omega', \square_1^*, x)| \leq Cp_k$. However just as in (346) one can show $|\Psi_{k,\Omega_{k+1}}(\Omega', \square_1^*, x) - \Psi_{k,\Omega_{k+1}}(\Omega', \square_2^*, x)| \leq e^{-r_{k+1}}$. This reduces the estimate to $|\partial \Psi_{k,\Omega_{k+1}}(\Omega', \square_2^*, x)| \leq Cp_k$, which we know since we control this derivative on $\tilde{\square}_2$.

Lemma 3.10. *The bracketed characteristic functions in (332) enforce the inequality on Ω_{k+1}*

$$|W_k| \leq Cp_k \quad (347)$$

Remark. A bound with $Cp_k \lambda_k^{-\frac{1}{4}}$ is easy. The issue is to eliminate the $\lambda_k^{-\frac{1}{4}}$.

Proof. First note that $\chi_k^q(\Omega_{k+1})$ is now saying that on Ω_{k+1}

$$\left| \Phi_{k+1} - Q \left(\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') + (C_{k,\Omega'}^{1/2})^{\text{loc}} W_k \right) \right| \leq p_k \quad (348)$$

By (339) this implies

$$\left| \Phi_{k+1} - Q \left(\Psi_{k,\Omega_{k+1}}(\Omega') + (C_{k,\Omega'}^{1/2})^{\text{loc}} W_k \right) \right| \leq 2p_k \quad (349)$$

However (344) can be rearranged to say

$$\Phi_{k+1} - Q \Psi_{k,\Omega_{k+1}}(\Omega') = \frac{a_k}{a_k + aL^{-2}} (\Phi_{k+1} - Q_{k+1} \phi_{k+1,\Omega'}^0) \quad (350)$$

This is bounded by Cp_k by (336). Thus we have on Ω_{k+1}

$$|Q(C_{k,\Omega'}^{1/2})^{\text{loc}} W_k| \leq Cp_k \quad (351)$$

We supplement this with a bound on $\partial(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k$. For this we look to the bounds of $\chi_k(\Omega_{k+1})$ which say that $(\Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W)$ is in $\mathcal{S}_k(\square)$ for any $\square \subset \Omega_{k+1}$. By lemma 3.2 this implies that on $\tilde{\square} \cap \Omega_{k+1}$

$$|\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W| \leq 2p_k\lambda_k^{-\frac{1}{4}} \quad |\partial(\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W)| \leq 3p_k \quad (352)$$

By the previous lemma $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}')$ alone satisfies these bounds with constants $Cp_k\lambda_k^{-\frac{1}{4}}, Cp_k$. Therefore on Ω_{k+1}

$$|(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k| \leq Cp_k\lambda_k^{-1/4} \quad |\partial(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k| \leq Cp_k \quad (353)$$

Now by (351) and the second bound in (353) we have

$$|(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k| \leq |(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k - Q(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k| + |Q(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k| \leq Cp_k \quad (354)$$

Finally the bound on W_k follows from this result and (306) which gives

$$|W_k| = \left| \left[(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} \right]^{-1} (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k \right| \leq C \|(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k\|_{\infty} \leq Cp_k \quad (355)$$

This completes the proof.

3.11.2 redundant characteristic functions

Now write the characteristic functions as

$$\left[C_{k,\mathbf{\Pi}} C_k^q(\Lambda_k, \Omega_{k+1}) \chi_k(\Lambda_k - \Lambda_{k+1}^{**}) \chi_k^q(\Omega_{k+1} - \Lambda_{k+1}^{**}) C_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) \right] \chi_k(\Lambda_{k+1}^{**}) \chi_k^q(\Lambda_{k+1}^{**}) \quad (356)$$

The bracketed characteristic functions here still enforce the bounds (333), (334). Indeed the bounds on Φ_k on $\delta\Omega_k$ and Φ_{k+1} on $\Omega_{k+1} - \Lambda_{k+1}^{**}$ are unaffected by the loss of $\chi_k(\Lambda_{k+1}^{**}) \chi_k^q(\Lambda_{k+1}^{**})$, and the new $C_{k+1}(\Omega_{k+1}, \Lambda_{k+1})$ supplies the even stronger bounds on Λ_{k+1}^{4*} :

$$|\Phi_{k+1}| \leq 2p_{k+1}\lambda_{k+1}^{-\frac{1}{4}}L^{-\frac{1}{2}} \quad |\partial\Phi_{k+1}| \leq 3p_{k+1}L^{-\frac{3}{2}} \quad (357)$$

We want to show that the last two characteristic functions in (356) are redundant. But first we show that the remaining terms in the bracket have no dependence on W_k in Λ_{k+1} . This is an important simplification for our fluctuation integral.

Lemma 3.11. $\chi_k(\Lambda_k - \Lambda_{k+1}^{**}) \chi_k^q(\Omega_{k+1} - \Lambda_{k+1}^{**})$ does not depend on W_k in Λ_{k+1} .

Proof. The Green's function $G_{k,\mathbf{\Omega}',r}(\square^*)$ only connects points in \square^* , thus $G_{k,\mathbf{\Omega}',r}(\square^*)f$ on \square only depends on f on \square^* . Hence $C_{k,\mathbf{\Omega}'}^{\frac{1}{2}}(\square^*)f$ on \square only depends on f in \square^* . It follows that $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}f$ on a set X only depends on f in X^* .

Now $\chi_k^q(\Omega_{k+1} - \Lambda_{k+1}^{**})$ depends on $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k$ in $(\Lambda_{k+1}^{**})^c$ and so on W_k in $((\Lambda_{k+1}^{**})^c)^* \subset (\Lambda_{k+1}^*)^c \subset \Lambda_{k+1}^c$. The function $\chi_k(\Lambda_k - \Lambda_{k+1}^{**})$ also depends on $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k$ in $(\Lambda_{k+1}^{**})^c$, but also on $\phi_{k,\mathbf{\Omega}(\square)}((C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k)$ on $\tilde{\square}$ for $\square \subset \Lambda_k - \Lambda_{k+1}^{**}$. By a similar argument this depends on W_k on $((\Lambda_{k+1}^{**})^c)^* \subset \Lambda_{k+1}^c$. This completes the proof.

Next a preliminary result:

Lemma 3.12. *The bracketed characteristic functions in (356) enforce the following inequalities*

1. *For an LM-cube \square in Ω_{k+1}^{\natural} we have on $\tilde{\square}$*

$$|\phi_{k+1, \Omega^+(\square)}^0 - \phi_{k+1, \Omega'}^0|, |\partial \phi_{k+1, \Omega^+(\square)}^0 - \partial \phi_{k+1, \Omega'}^0| \leq CM^{-\frac{1}{2}} p_{k+1} \quad (358)$$

2. *For an M cubes \square in Ω_{k+1}^{\natural} let $\phi_{k, \Omega(\square)}^{\min} = \phi_{k, \Omega(\square)}(\tilde{Q}_{\mathbb{T}^0, \Omega(\square)}^T \Psi_{k, \Omega_{k+1}}(\Omega'))$ (i.e. we replace $\Phi_{k, \Omega_{k+1}}$ with the minimizer). Then on $\tilde{\square}$*

$$|\phi_{k, \Omega(\square)}^{\min} - \phi_{k, \Omega(\Lambda_k^*)}(\hat{\Psi}_{k, \Omega'})|, |\partial \phi_{k, \Omega(\square)}^{\min} - \partial \phi_{k, \Omega(\Lambda_k^*)}(\hat{\Psi}_{k, \Omega'})| \leq CM^{-\frac{1}{2}} p_k \quad (359)$$

Proof. Straightforward estimates would give an unwanted factor $\lambda_{k+1}^{-\frac{1}{4}}$ so we must take another path. First we note that we can replace $\phi_{k+1, \Omega'}^0 = \phi_{k+1, \Omega'}^0(\hat{\Phi}_{k+1, \Omega'})$ by $\phi_{k+1, \Omega'}^0 = \phi_{k+1, \Omega'}^0(0, \Phi_{k+1})$. This is so since the difference $\hat{\Phi}_{k+1, \Omega'} - (0, \Phi_{k+1, \Omega_{k+1}})$ is localized in Ω_{k+1}^c and so is $[r_{k+1}]$ LM-cubes away from Ω_{k+1}^{\natural} . Thus a straightforward estimate generates a factor $e^{-r_{k+1}}$ which is enough to dominate a $\lambda_{k+1}^{-\frac{1}{4}}$ and an $M^{\frac{1}{2}}$.

Next we use the representation developed in lemma 3.1, now in the prescaled version. Let Δ_y be an L -cube in $\tilde{\square}$. Then for x in a neighborhood of Δ_y as in (183):

$$\begin{aligned} \left[\phi_{k+1, \Omega^+(\square)}^0 \left(\tilde{Q}_{\mathbb{T}^1, \Omega^+(\square)}^T \Phi_{k+1} \right) \right](x) &= \left[\phi_{k+1, \Omega^+(\square)}^0 \left(\tilde{Q}_{\mathbb{T}^1, \Omega^+(\square)}^T (\Phi_{k+1} - \Phi_{k+1}(y)) \right) \right](x) \\ &\quad + \left[1 - \bar{\mu}_k G_{k+1, \Omega^+(\square)}^0 \cdot 1 \right](x) \Phi_{k+1}(y) \\ \left[\phi_{k+1, \Omega'}^0 \left(0, \Phi_{k+1} \right) \right](x) &= \left[\phi_{k+1, \Omega'}^0 \left(0, \Phi_{k+1} - \Phi_{k+1}(y) \right) \right](x) \\ &\quad + \left[1 - \bar{\mu}_k G_{k+1, \Omega'}^0 \cdot (0, 1_{\Omega_{k+1}}) \right](x) \Phi_{k+1}(y) \end{aligned} \quad (360)$$

As in lemma 3.1 we can use the bound (357) on $\partial \Phi_{k+1}$ to estimate the the first term in (360) by

$$\begin{aligned} &\sum_{y'} \left| \phi_{k+1, \Omega^+(\square)}^0 \left(1_{\Delta_{y'}} \tilde{Q}_{\mathbb{T}^1, \Omega^+(\square)}^T (\Phi_{k+1} - \Phi_{k+1}(y)) \right) (x) \right| \\ &\leq \sum_{y'} C e^{-\frac{1}{4} \gamma_0 d_{\Omega^+(\square)}(y, y')} \|1_{\Delta_{y'}} Q_{\mathbb{T}^1, \Omega^+(\square)}^T (\Phi_{k+1} - \Phi_{k+1}(y))\|_{\infty} \\ &\leq \sum_{y'} C e^{-\frac{1}{4} \gamma_0 d_{\Omega^+(\square)}(y, y')} (d(y, y') + 1) p_{k+1} \leq C p_{k+1} \end{aligned} \quad (361)$$

We need a smaller constant here. This occurs in terms coming from $\Delta_{y'}$ not in $\square^{\sim 2}$. This is because in the random walk expansion the Green's functions must have at least one step to get from $\tilde{\square}$ to $\square^{\sim 2}$. This enables us to extract a factor $M^{-\frac{1}{2}}$ without spoiling the convergence of the expansion. Thus we get the bounds $CM^{-\frac{1}{2}} p_{k+1}$. The same works for an estimate on these distant terms in $\phi_{k+1, \Omega'}^0$.

Now consider terms with $\Delta_y \subset \square^{\sim 2}$. In this case the difference of the two first terms in (360) is

$$\sum_{y' \subset \square^{\sim 2}} \left[\left(G_{k+1, \Omega^+(\square)}^0 - G_{k+1, \Omega'}^0 \right) Q_{k+1}^T L^{-2} a_{k+1} \left(1_{\Delta_{y'}} (\Phi_{k+1} - \Phi_{k+1}(y)) \right) \right](x) \quad (362)$$

This reduces the problem to an estimate on the difference of the Green's functions. We claim that for $y, y' \in \square^{\sim 2}$

$$|1_{\Delta_y} \left(G_{k+1, \Omega^+(\square)}^0 - G_{k+1, \Omega'}^0 \right) 1_{\Delta_{y'}} f| \leq CM^{-\frac{1}{2}} e^{-\mathcal{O}(1)L^{-2}d(y, y')} \|f\|_{\infty} \quad (363)$$

This is so since $[G_{k+1, \mathbf{\Omega}^+(\square)}^0]^{-1}$ and $[G_{k+1, \mathbf{\Omega}'}^0]^{-1}$ agree on a set containing $\square^{\sim 2}$. Thus in a random walk expansion connecting points in $\square^{\sim 2}$ they have the same leading term which cancels. The remaining terms have more than one step and supply the factor $M^{-\frac{1}{2}}$. Again we have a bound $CM^{-\frac{1}{2}}p_{k+1}$,

Finally consider the difference of the last two terms in (360) which is

$$\bar{\mu}_k \left[(G_{k+1, \mathbf{\Omega}^+(\square)}^0 \cdot 1 - G_{k+1, \mathbf{\Omega}'}^0 \cdot (0, 1_{\Omega_{k+1}})) \right] (x) \Phi_{k+1}(y) \quad (364)$$

We can replace the $(0, 1_{\Omega_{k+1}})$ here by 1 since the difference is $\mathcal{O}(e^{-r_{k+1}})$ and is completely negligible. Then we must estimate

$$\bar{\mu}_k \left[\left(G_{k+1, \mathbf{\Omega}^+(\square)}^0 - G_{k+1, \mathbf{\Omega}'}^0 \right) \cdot 1 \right] (x) \Phi_{k+1}(y) \quad (365)$$

Again we first localize in $\square^{\sim 2}$ and then use the estimate (363). Then this term is bounded by $\bar{\mu}_k (CM^{-\frac{1}{2}}p_{k+1}) \lambda_{k+1}^{-\frac{1}{4}}$. Since $\bar{\mu}_k \lambda_{k+1}^{-\frac{1}{4}} \leq \mathcal{O}(1)$ we again have the bound $CM^{-\frac{1}{2}}p_{k+1}$.

This completes the estimate on $\phi_{k+1, \mathbf{\Omega}^+(\square)}^0 - \phi_{k+1, \mathbf{\Omega}'}^0$. The estimate on the derivatives follows in just the same way, now using estimates on the derivatives of the Green's functions.

The proof of the estimates on $\phi_{k, \mathbf{\Omega}(\square)}^{\min} - \phi_{k, \mathbf{\Omega}(\Lambda_k^*)}(\hat{\Psi}_{k, \mathbf{\Omega}'})$ also follows in the same way. Here the relevant input is the bounds (338) on $\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}')$. This completes the proof.

Lemma 3.13. *The bracketed characteristic functions in (356) enforce*

$$\chi_k(\Lambda_{k+1}^{**}) \chi_k^q(\Lambda_{k+1}^{**}) = 1 \quad (366)$$

Proof. To show $\chi_k(\Lambda_{k+1}^{**}) = 1$ we must show that $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k \in \mathcal{S}_k(\square)$ for any M -cube $\square \subset \Lambda_{k+1}^{**}$. We argue separately that $(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k \in \frac{1}{2} \mathcal{S}_k(\square)$ and that $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') \in \frac{1}{2} \mathcal{S}_k(\square)$.

For the $(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$ bounds start with the fact that $|W_k| \leq p_{0,k}$ on Λ_k^{4*} . Since $(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$ on Λ_k^{3*} depends on W_k on Λ_k^{4*} we have on Λ_k^{3*} by (305)

$$|(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k| \leq Cp_{0,k} \quad |\partial(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k| \leq Cp_{0,k} \quad (367)$$

The second bound follows from the first since we are on a unit lattice. Then $(Cp_{0,k})^{-1} p_k (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$ satisfies the same bounds with only a p_k on the right side. By lemma 3.1 for $\square \subset \Lambda_k^{**}$ we have $(Cp_{0,k})^{-1} p_k (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k \in C\mathcal{S}_k(\square)$ or $(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k \in Cp_{0,k} p_k^{-1} \mathcal{S}_k(\tilde{\square})$. But for $p_0 < p$ and λ_k sufficiently small $Cp_{0,k} p_k^{-1} = C(-\log \lambda_k)^{p_0-p} < \frac{1}{2}$ so we have the result.

For the $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}')$ bounds recall that for an LM cube $\square \subset \Lambda_{k+1}^{4*}$, Φ_{k+1} satisfies the bounds (322) on $\tilde{\square}$. For M sufficiently large $CM^{-\frac{1}{2}}p_{k+1}$ is smaller than anything on the right side of these equations. Thus by (358) we may replace $\phi_{k+1, \mathbf{\Omega}(\square)}^0$ by $\phi_{k+1, \mathbf{\Omega}'}^0$. Then for $\square \subset \Lambda_{k+1}^{4*}$ on $\tilde{\square}$:

$$\begin{aligned} |\Phi_{k+1} - Q_{k+1} \phi_{k+1, \mathbf{\Omega}'}^0| &\leq 2p_{k+1} L^{-\frac{1}{2}} \\ |\partial \phi_{k+1, \mathbf{\Omega}'}^0| &\leq 2p_{k+1} L^{-\frac{3}{2}} \\ |\phi_{k+1, \mathbf{\Omega}'}^0| &\leq 2p_{k+1} \lambda_{k+1}^{-1/4} L^{-\frac{1}{2}} \end{aligned} \quad (368)$$

Next recall that

$$\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}') = Q_k \phi_{k+1, \mathbf{\Omega}'}^0 + \frac{aL^{-2}}{a_k + aL^{-2}} Q^T (\Phi_{k+1} - Q_{k+1} \phi_{k+1, \mathbf{\Omega}'}^0) \quad (369)$$

This lets us replace $\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega'}^0$ by $\Psi_{k,\Omega_{k+1}}(\Omega') - Q_k\phi_{k+1,\Omega'}^0$ in the above inequality. Furthermore in (252) we have already noted the identity $\phi_{k+1,\Omega'}^0 = \phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})$. Thus the above inequalities (368) become on $(\Lambda_{k+1}^{4*})^\sim$

$$\begin{aligned} |\Psi_{k,\Omega_{k+1}}(\Omega') - Q_k\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})| &\leq 2p_{k+1}L^{-\frac{1}{2}} \\ |\partial\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})| &\leq 2p_{k+1}L^{-\frac{3}{2}} \\ |\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})| &\leq 2p_{k+1}\lambda_{k+1}^{-1/4}L^{-\frac{1}{2}} \end{aligned} \quad (370)$$

Now for $\square \subset \Lambda_{k+1}^{**}$ (359) lets us replace $\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})$ by $\phi_{k,\Omega(\square)}^{\min} = \phi_{k,\Omega(\square)}(\tilde{Q}_{T_0,\Omega(\square)}^T \Psi_{k,\Omega_{k+1}}(\Omega'))$ for M sufficiently large, and so on $\tilde{\square}$

$$\begin{aligned} |\Psi_{k,\Omega_{k+1}}(\Omega') - Q_k\phi_{k,\Omega(\square)}^{\min}| &\leq 3p_{k+1}L^{-\frac{1}{2}} \\ |\partial\phi_{k,\Omega(\square)}^{\min}| &\leq 3p_{k+1}L^{-\frac{3}{2}} \\ |\phi_{k,\Omega(\square)}^{\min}| &\leq 3p_{k+1}\lambda_{k+1}^{-1/4}L^{-\frac{1}{2}} \end{aligned} \quad (371)$$

Since $\lambda_{k+1}^{-\frac{1}{4}} < \lambda_k^{-\frac{1}{4}}$ and $p_{k+1} \leq (1 + \log L)^p p_k$ this says that $\Psi_{k,\Omega_{k+1}}(\Omega') \in \frac{1}{4}\mathcal{S}_k(\square)$ for L sufficiently large. But $|\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') - \Psi_{k,\Omega_{k+1}}(\Omega')| \leq e^{-r_{k+1}}$ from (339). Hence $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') \in \frac{1}{2}\mathcal{S}_k(\square)$. This completes the proof that $\chi_k(\Lambda_{k+1}^{**}) = 1$.

Now we show $\chi_k^q(\Lambda_{k+1}^{**}) = 1$, that is we show that on $\square \subset \Lambda_{k+1}^{**}$

$$|\Phi_{k+1} - Q(\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') + (C_{k,\Omega}^{1/2})^{\text{loc}}W)| \leq p_k \quad (372)$$

By (339) and (367) this reduces to showing that $|\Phi_{k+1} - Q\Psi_{k,\Omega_{k+1}}(\Omega')| \leq \frac{1}{2}p_k$. This follows by (350) and then (368):

$$|\Phi_{k+1} - Q\Psi_{k,\Omega_{k+1}}(\Omega')| \leq |\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega'}^0| \leq 2p_{k+1}L^{-\frac{1}{2}} \leq \frac{1}{2}p_k \quad (373)$$

3.11.3 bounds for analyticity domains

In an expression like $R_{k,\mathbf{II}}(\Lambda_k, \Phi_{k,\Lambda_k - \Omega_{k+1}}\Psi_{k,\Omega'}^{\text{loc}} + (C_{k,\Omega'}^{1/2})^{\text{loc}}W_k)$, we need to know how large we can allow the fundamental fields Φ_k, Φ_{k+1} to be, and still stay in the region of analyticity $\mathcal{P}_k(\Lambda_k, 2\delta)$ for $R_{k,\mathbf{II}}(\Lambda_k)$. The following is a result in that direction. The proof is similar to lemma 3.13.

Define for an LM -cube $\square \subset \Omega_{k+1}$ the domain $\mathcal{P}_{k+1}^0(\square, \delta)$ to be all fields satisfying

$$\begin{aligned} |\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega^+(\square)}| &\leq \lambda_{k+1}^{-\frac{1}{4}-\delta}L^{-\frac{1}{2}} \quad \text{on } \tilde{\square} \cap \Omega_{k+1} \\ |\partial\phi_{k+1,\Omega^+(\square)}| &\leq \lambda_{k+1}^{-\frac{1}{4}-\delta}L^{-\frac{3}{2}} \quad \text{on } \tilde{\square} \\ |\phi_{k+1,\Omega^+(\square)}| &\leq \lambda_{k+1}^{-\frac{1}{4}-\delta}L^{-\frac{1}{2}} \quad \text{on } \tilde{\square} \end{aligned} \quad (374)$$

For $X \subset \Omega_{k+1}$ define

$$\mathcal{P}_{k+1}^0(X, \delta) = \bigcap_{\square \subset X} \mathcal{P}_{k+1}^0(\square, \delta) \quad (375)$$

Lemma 3.14. *If*

$$(\Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}}) \in \mathcal{P}'_k(\Omega_k^{\natural} - \Omega_{k+1}, \delta) \cap \mathcal{P}_{k+1}^0(\Omega_{k+1} - \Omega_{k+1}^{2\natural}, \delta) \cap \mathcal{P}_{k+1}^0(\Omega_{k+1}^{2\natural}, 2\delta) \quad (376)$$

then

$$\left(\Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') \right) \in \frac{1}{2} \mathcal{P}_k(\Omega_k^{\natural}, 2\delta) \quad (377)$$

Remarks.

1. On the domain (376) we have

$$\begin{aligned} |\Phi_k| &\leq 2\lambda_k^{-\frac{1}{4}-\delta} \text{ on } \Omega_k^{\natural} - \Omega_{k+1} \\ |\Phi_{k+1}| &\leq 2\lambda_{k+1}^{-\frac{1}{4}-\delta} L^{-\frac{1}{2}} \text{ on } \Omega_{k+1} - \Omega_{k+1}^{2\natural} \\ |\Phi_{k+1}| &\leq 2\lambda_{k+1}^{-\frac{1}{4}-2\delta} L^{-\frac{1}{2}} \text{ on } \Omega_{k+1}^{2\natural} \end{aligned} \quad (378)$$

We choose (376) because it contains the domain

$$\mathcal{P}_{k+1,\Omega}^0 \equiv \bigcap_{j=1}^k [\mathcal{P}'_j(\delta\Omega_j, \delta)]_{L^{-(k-j)}} \cap \mathcal{P}_{k+1}^0(\Omega_{k+1} - \Omega_{k+1}^{2\natural}, \delta) \cap \mathcal{P}_{k+1}^0(\Omega_{k+1}^{2\natural}, 2\delta) \quad (379)$$

which scales to $\mathcal{P}_{k+1,\Omega'}$. Note also that $\mathcal{P}_k(\Omega_k^{\natural}, 2\delta)$ is contained in $\mathcal{P}_k(\Lambda_k, 2\delta)$.

2. The domain suffers some shrinkage in $\Omega_k^{\natural} - \Omega_{k+1}^{2\natural}$ (we need δ to get 2δ), but not in $\Omega_{k+1}^{2\natural}$. The shrinkage in $\Omega_k^{\natural} - \Omega_{k+1}^{2\natural}$ is tolerable since it is not repeated. Shrinkage in $\Omega_{k+1}^{2\natural}$ might be repeated and eventually would lead to problems.

Proof. For an M -cube $\square \subset \Omega_k^{\natural}$, we must show that $(\Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega')) \in \frac{1}{2} \mathcal{P}_k(\square, 2\delta)$. We distinguish several cases.

(A.) $\square \subset \Omega_{k+1}^{\natural}$. Start by considering a LM -cube $\square \subset \Omega_{k+1}^{\natural}$. Our assumptions imply that on $\tilde{\square}$ (LM -cube enlargement)

$$|\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega^+(\square)}^0|, |\partial\phi_{k+1,\Omega^+(\square)}^0|, |\phi_{k+1,\Omega^+(\square)}^0| \leq \lambda_{k+1}^{-\frac{1}{4}-2\delta} L^{-\frac{1}{2}} \quad (380)$$

where $\phi_{k+1,\Omega^+(\square)}^0$ only depends on Φ_{k+1} . We claim that on $\tilde{\square}$

$$|\phi_{k+1,\Omega(\square)}^0 - \phi_{k+1,\Omega'}^0|, |\partial\phi_{k+1,\Omega(\square)}^0 - \partial\phi_{k+1,\Omega'}^0| \leq CM^{-\frac{1}{2}} \lambda_{k+1}^{-\frac{1}{4}-2\delta} \quad (381)$$

This follows by a variation of the argument leading to (358), now with the larger bounds (378) on the fields. Since fields and derivatives have the same weight the identities (360) are not required here. The main idea is to split the contribution of Φ_{k+1} into a piece not in $\square^{\sim 2}$ where the fields individually have the claimed bound, and a piece in $\square^{\sim 2}$ where the difference of the Green's functions supplies the factor $M^{-\frac{1}{2}}$.

Given (381), then for M large we can replace $\phi_{k+1,\Omega^+(\square)}^0$ by $\phi_{k+1,\Omega'}^0$ in (380) and have on $(\Omega_{k+1}^{\natural})^{\sim}$

$$|\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega'}^0|, |\partial\phi_{k+1,\Omega'}^0|, |\phi_{k+1,\Omega'}^0| \leq 2\lambda_{k+1}^{-\frac{1}{4}-2\delta} L^{-\frac{1}{2}} \quad (382)$$

Now replace $|\Phi_{k+1} - Q_{k+1}\phi_{k+1,\Omega'}^0|$ by the smaller $|\Psi_{k,\Omega_{k+1}}(\Omega') - Q_k\phi_{k+1,\Omega'}^0|$ and use the identity $\phi_{k+1,\Omega'}^0 = \phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})$ which still holds with complex fields. This yields on $(\Omega_{k+1}^{\natural})^\sim$

$$|\Psi_{k,\Omega_{k+1}}(\Omega') - Q_k\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})|, |\partial\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})|, |\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})| \leq 2\lambda_{k+1}^{-\frac{1}{4}-2\delta}L^{-\frac{1}{2}} \quad (383)$$

The bounds (378) imply that $|\phi_{k+1,\Omega'}^0| \leq C\lambda_{k+1}^{-\frac{1}{4}-2\delta}$ and it follows that $|\Psi_{k,\Omega_{k+1}}(\Omega')| \leq C\lambda_{k+1}^{-\frac{1}{4}-2\delta}$ on its full domain. Then if \square is an M -cube in Ω_{k+1}^{\natural} , by a variation of (359) we have (again no special identity required) on $\tilde{\square}$ (M -cube enlargement):

$$|\phi_{k,\Omega(\square)}^{\min} - \phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})|, |\partial\phi_{k,\Omega(\square)}^{\min} - \partial\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})| \leq CM^{-\frac{1}{2}}\lambda_{k+1}^{-\frac{1}{4}-2\delta} \quad (384)$$

Hence for M large we can replace $\phi_{k,\Omega(\Lambda_k^*)}(\hat{\Psi}_{k,\Omega'})$ by $\phi_{k,\Omega(\square)}^{\min}$ in (383) and obtain on $\tilde{\square}$

$$|\Psi_{k,\Omega_{k+1}}(\Omega') - Q_k\phi_{k,\Omega(\square)}^{\min}|, |\partial\phi_{k,\Omega(\square)}^{\min}|, |\phi_{k,\Omega(\square)}^{\min}| \leq 3\lambda_{k+1}^{-\frac{1}{4}-2\delta}L^{-\frac{1}{2}} \quad (385)$$

The same holds with $\hat{\Psi}_{k,\Omega'}$ replaced by $\hat{\Psi}_{k,\Omega'}^{\text{loc}}$ and $4\lambda_{k+1}^{-\frac{1}{4}-2\delta}L^{-\frac{1}{2}}$ on the right side. But for L large, $4\lambda_{k+1}^{-\frac{1}{4}-2\delta}L^{-\frac{1}{2}} = 4\lambda_k^{-\frac{1}{4}-2\delta}L^{-\frac{3}{4}-2\delta} \leq \frac{1}{2}\lambda_k^{-\frac{1}{4}-2\delta}$. Thus we have $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega') \in \frac{1}{2}\mathcal{P}_k(\square, 2\delta)$ as required.

(B.) $\square \subset \Omega_{k+1} - \Omega_{k+1}^{\natural}$. This time we take a more direct approach. The bounds (378) imply that on $[(\Omega_{k+1}^{\natural})^c]^\sim$ (LM -cube enlargement) we have $|\phi_{k+1,\Omega'}^0| \leq C\lambda_{k+1}^{-\frac{1}{4}-\delta}$. The point here is that the weaker bound $\mathcal{O}(\lambda_{k+1}^{-\frac{1}{4}-2\delta})$ in $\Omega_{k+1}^{2\natural}$ is offset by a factor $e^{-r_{k+1}}$ due to the distance between $[(\Omega_{k+1}^{\natural})^c]^\sim$ and $\Omega_{k+1}^{2\natural}$. It follows that on $[(\Omega_{k+1}^{\natural})^c]^\sim \cap \Omega_{k+1}$ we have

$$|\Psi_{k,\Omega_{k+1}}(\Omega')| \leq C\lambda_{k+1}^{-\frac{1}{4}-\delta} \quad (386)$$

Let \square be an M -cube in $\Omega_{k+1} - \Omega_{k+1}^{\natural}$, and now $\phi_{k,\Omega(\square)}^{\min} = \phi_{k,\Omega(\square)}(\tilde{Q}_{\mathbb{T}_0,\Omega(\square)}^T(\Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}(\Omega')))$. We claim that on $\tilde{\square}$ (M -cube enlargement)

$$|\Psi_{k,\Omega_{k+1}}(\Omega') - Q_k\phi_{k,\Omega(\square)}^{\min}|, |\partial\phi_{k,\Omega(\square)}^{\min}|, |\phi_{k,\Omega(\square)}^{\min}| \leq C\lambda_{k+1}^{-\frac{1}{4}-\delta} \quad (387)$$

For example to bound $|\phi_{k,\Omega(\square)}^{\min}|$ we need a bound on $\Psi_{k,\Omega_{k+1}}(\Omega')$ on $\square^{\sim((2R+1))} \cap \Omega_{k+1}$. But this is included in the LM enlargement in (386), assuming $L > 2R + 1$. A bound of the same form holds with $\Psi_{k,\Omega_{k+1}}(\Omega')$ replaced by $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\Omega')$. This gives the result since for λ_k sufficiently small $C\lambda_{k+1}^{-\frac{1}{4}-\delta} \leq \frac{1}{2}\lambda_k^{-\frac{1}{4}-2\delta}$

(C.) $\square \subset \Omega_k^{\natural} - \Omega_{k+1}$. Argue as in the previous case. Note that now we are trying to prove on $\tilde{\square}$

$$|\Phi_k - Q_k\phi_{k,\Omega(\square)}^{\min}|, |\partial\phi_{k,\Omega(\square)}^{\min}|, |\phi_{k,\Omega(\square)}^{\min}| \leq \frac{1}{2}\lambda_{k+1}^{-\frac{1}{4}-2\delta} \quad (388)$$

3.12 adjustments

Consider the definition (329) of $\mathcal{C}_{k+1}(\Omega_{k+1}, \Lambda_{k+1})$. Since $\Lambda_{k+1} \subset \Omega_{k+1}^\natural - Q_{k+1}$ and $\Lambda_{k+1} \subset \Omega_{k+1} - R_{k+1}$, we can extract $\chi_{k+1}^0(\Lambda_{k+1})$ and $\chi_k^w(\Lambda_{k+1})$ from each term in this sum and write

$$\mathcal{C}_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) = \chi_{k+1}^0(\Lambda_{k+1}) \chi_k^w(\Lambda_{k+1}) \mathcal{C}'_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) \quad (389)$$

where

$$\begin{aligned} & \mathcal{C}'_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) \\ = & \sum_{Q_{k+1}, R_{k+1} \rightarrow \Lambda_{k+1}} \zeta_{k+1}^0(Q_{k+1}) \zeta_k^w(R_{k+1}) \chi_{k+1}^0(\Omega_{k+1}^\natural - (Q_{k+1} \cup \Lambda_{k+1})) \chi_k^w(\Omega_{k+1} - (R_{k+1} \cup \Lambda_{k+1})) \end{aligned} \quad (390)$$

Using lemma 3.13 the characteristic functions now have the form

$$\mathcal{C}_{k+1, \mathbf{\Pi}^+}^0 \chi_{k+1}^0(\Lambda_{k+1}) \chi_k^w(\Lambda_{k+1}) \quad (391)$$

where $\mathcal{C}_{k+1, \mathbf{\Pi}^+}^0 = \mathcal{C}_{k, \mathbf{\Pi}} \mathcal{C}_{k+1, \Lambda_k, \Omega_{k+1}, \Lambda_{k+1}}^0$ and

$$\mathcal{C}_{k+1, \Lambda_k, \Omega_{k+1}, \Lambda_{k+1}}^0 = \mathcal{C}_k^q(\Lambda_k, \Omega_{k+1}) \chi_k(\Lambda_k - \Lambda_{k+1}^{**}) \chi_k^q(\Omega_{k+1} - \Lambda_{k+1}^{**}) \mathcal{C}'_{k+1}(\Omega_{k+1}, \Lambda_{k+1}) \quad (392)$$

Then $\mathcal{C}_{k+1, \Lambda_k, \Omega_{k+1}, \Lambda_{k+1}}^0$ does not depend on $\Phi_{k+1, \Lambda_{k+1}^\natural}$. The function $\mathcal{C}_{k+1, \mathbf{\Pi}^+}^0$ enforces on $\Lambda_k - \Omega_{k+1}$ (due to $\chi_k(\Lambda_k - \Lambda_{k+1}^{**})$)

$$|\Phi_k| \leq 2p_k \lambda_k^{-\frac{1}{4}} \quad |\partial \Phi_k| \leq 3p_k \quad (393)$$

and on $\Omega_{k+1} - \Lambda_{k+1}$ by (334) and (347)

$$|\Phi_{k+1}| \leq 3p_k \lambda_k^{-1/4} \quad |\partial \Phi_{k+1}| \leq 4p_k \quad |W_k| \leq Cp_k \quad (394)$$

The function $\mathcal{C}_{k+1, \mathbf{\Pi}^+}$ will be $\mathcal{C}_{k+1, \mathbf{\Pi}^+}^0$ scaled down and the bounds (393), (394) scale to the required bounds (206), (207) for $k+1$, provided $C_w > C$.

In the expression (331) for $\tilde{\rho}_{k+1}(\Phi_{k+1})$ we split the fluctuation measure by

$$d\mu_{\Omega_{k+1}}(W_k) = d\mu_{\Omega_{k+1} - \Lambda_{k+1}}(W_k) d\mu_{\Lambda_{k+1}}(W_k) \quad (395)$$

With the first factor we form $dW_{k+1, \mathbf{\Pi}^+}^0 \equiv dW_{k, \mathbf{\Pi}} d\mu_{\Omega_{k+1} - \Lambda_{k+1}}(W_k)$. For the second factor we introduce the probability measure

$$d\mu_{\Lambda_{k+1}}^*(W_k) = (\mathcal{N}_{k, \Lambda_{k+1}}^w)^{-1} \chi_k^w(\Lambda_{k+1}) d\mu_{\Lambda_{k+1}}(W_k) \quad (396)$$

Since the measure is a product over sites in $\Lambda_{k+1}^{(k)}$, the normalization factor can be written

$$\mathcal{N}_{k, \Lambda_{k+1}}^w = \int \chi_k^w(\Lambda_{k+1}) d\mu_{\Lambda_{k+1}}(W_k) = \exp(-\varepsilon_k^0 |\Lambda_{k+1}^{(k)}|) = \exp(-\varepsilon_k^0 \text{Vol}(\Lambda_{k+1})) \quad (397)$$

This defines ε_k^0 which is the same as in part I.

Now (331) becomes

$$\begin{aligned} \tilde{\rho}_{k+1}(\Phi_{k+1}) = & Z_{k+1}^0 \sum_{\mathbf{\Pi}^+} \int d\Phi_{k+1, \mathbf{\Omega}^{+,c}}^0 dW_{k+1, \mathbf{\Pi}^+}^0 K_{k, \mathbf{\Pi}} \mathcal{C}_{k+1, \mathbf{\Pi}^+}^0 \exp\left(c_{k+1} |\Omega_{k+1}^{c, (k)}|\right) \\ & \chi_{k+1}^0(\Lambda_{k+1}) \exp\left(-S_{k+1}^{*, 0}(\Lambda_k)\right) \Xi_{k, \mathbf{\Pi}^+} \end{aligned} \quad (398)$$

Here we have defined the fluctuation integral

$$\Xi_{k,\mathbf{\Pi}^+} = \exp(-\varepsilon_k^0 \text{Vol}(\Lambda_{k+1})) \int d\mu_{\Lambda_{k+1}}^*(W_k) \exp\left(E_k^+(\Lambda_k) + R_{\mathbf{\Pi},\Omega_{k+1}}^{(\leq 7)} + B_{k,\mathbf{\Pi}}(\Lambda_k)\right) \quad (399)$$

We further define $\delta E_k^+(X, \phi, \mathcal{W})$ by

$$E_k^+(X, \phi + \mathcal{W}) = E_k^+(X, \phi) + \delta E_k^+(X, \phi, \mathcal{W}) \quad (400)$$

This is the same as the definition in part I, but here we have different fields $\phi = \phi_{k+1,\mathbf{\Omega}'}^0$ and $\mathcal{W} = \mathcal{W}_{k,\mathbf{\Omega}'}$. The factor $E_k^+(\Lambda_k, \phi_{k+1,\mathbf{\Omega}'}^0)$ does not depend on W_k and can be moved outside the integral. Inside the integral we have $\delta E_k^+(\Lambda_k) = \delta E_k^+(\Lambda_k, \phi_{k+1,\mathbf{\Omega}'}^0, \mathcal{W}_{k,\mathbf{\Omega}'})$ and with these adjustments

$$\Xi_{k,\mathbf{\Pi}^+} = \exp\left(-\varepsilon_k^0 \text{Vol}(\Lambda_{k+1}) + E_k^+(\Lambda_k, \phi_{k+1,\mathbf{\Omega}'}^0)\right) \Xi'_{k,\mathbf{\Pi}^+} \quad (401)$$

where

$$\Xi'_{k,\mathbf{\Pi}^+} = \int d\mu_{\Lambda_{k+1}}^*(W_k) \exp\left(\delta E_k^+(\Lambda_k) + R_{\mathbf{\Pi},\Omega_{k+1}}^{(\leq 7)} + B_{k,\mathbf{\Pi}}(\Lambda_k)\right) \quad (402)$$

3.13 localization

We will be giving a local structure to the fluctuation integral by a cluster expansion. As input to this we give localization expansions for the integrand.

We start with $\delta E_k^+(\Lambda_k) = \sum_{X \subset \Lambda_k} \delta E_k^+(X)$. The function $\delta E_k^+(X, \phi_{k+1,\mathbf{\Omega}'}^0, \mathcal{W}_{k,\mathbf{\Omega}'})$ depends on $\mathcal{W}_{k,\mathbf{\Omega}'} = a_k G_{k,\mathbf{\Omega}(\Lambda_k^*)} Q_k^T(C_k^{1/2}, W_k)$ only in X , but depends on W_k in all of $\Omega'_1 = \Omega_1(\Lambda_k^*)$. We need a sharper localization in W_k .

Lemma 3.15. *For $(\Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}})$ in (376) (hence satisfying (378)), and $|W_k| \leq B_w p_k$:*

$$\begin{aligned} \delta E_k^+(\Lambda_k, \phi_{k+1,\mathbf{\Omega}'}^0, \mathcal{W}_{k,\mathbf{\Omega}'}) &= \sum_{Y \in \mathcal{D}_{k+1}^0, Y \subset \Lambda_{k+1}} (\delta E_k^+)^{\text{loc}}(Y, \phi_{k+1,\mathbf{\Omega}'}^0, W_k) \\ &+ \sum_{Y \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c), Y \# \Lambda_{k+1}} B_{k,\mathbf{\Pi}^+}^{(E)}(Y) + \tilde{B}_{k+1,\mathbf{\Pi}^+} \text{ terms} \end{aligned} \quad (403)$$

where

1. For $Y \in \mathcal{D}_{k+1}^0$ the leading term $(\delta E_k^+)^{\text{loc}}(Y, \phi, W_k)$ is exactly the global small field expression from part I. It depends on ϕ, W_k only in Y , is analytic in $\phi \in \frac{1}{2}\mathcal{R}_k$ and $|W_k| \leq B_w p_k$ and satisfies there

$$|(\delta E_k^+)^{\text{loc}}(Y, \phi, W_k)| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-2\kappa_0-2)d_{LM}(Y)} \quad (404)$$

2. For $Y \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c)$ the boundary term $B_{k,\mathbf{\Pi}^+}^{(E)}(Y, \Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}}, W_k)$ depends on the fields only in Y . It is analytic the stated domain and satisfies there

$$|B_{k,\mathbf{\Pi}^+}^{(E)}(Y)| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-2\kappa_0-3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \quad (405)$$

Remarks.

1. The expression “ $\tilde{B}_{k+1, \mathbf{\Pi}^+}$ terms” will be used repeatedly. It refers to functions localized in Λ_{k+1}^c which are bounded by $C|\Lambda_k^{(k)} - \Lambda_{k+1}^{(k)}| = C \text{Vol}(\Lambda_k - \Lambda_{k+1})$. Local structure is no longer important for these terms.
2. The bounds (142) and (378) and $2\delta < \epsilon$ yield $|\phi_{k+1, \mathbf{\Omega}'}^0| \leq C\lambda_k^{-\frac{1}{4}-2\delta} \leq \frac{1}{2}\lambda_k^{-\frac{1}{4}-3\epsilon}$ with similar bounds on derivatives. Thus $\phi_{k+1, \mathbf{\Omega}'}^0$ is in $\frac{1}{2}\mathcal{R}_k$ as required.

Note also that (133) and (305) and (378) show that $\mathcal{W}_{k, \mathbf{\Omega}'} = \phi_{k, \mathbf{\Omega}(\Lambda_k^*)}(C_{k, \mathbf{\Omega}'}^{1/2} W_k)$ satisfies $|\mathcal{W}_{k, \mathbf{\Omega}'}| \leq C\|C_{k, \mathbf{\Omega}'}^{1/2} W_k\|_\infty \leq CB_w p_k$.

Proof. (A.) We study $\delta E_k^+(X, \phi, \mathcal{W}_{k, \mathbf{\Omega}'})$ for $\phi \in \frac{1}{2}\mathcal{R}_k$. We argue as in lemma 17 of Part I. There are two parts $\delta E_k^+ = \delta E_k - \delta V_k$. The potential is supported on cubes \square and has the form $\delta V_k(\square, \phi, \mathcal{W}_{k, \mathbf{\Omega}'}) = \delta V_k(\square, \phi + \mathcal{W}_{k, \mathbf{\Omega}'}) - \delta V_k(\square, \phi)$. The bounds $|\phi| \leq \lambda_k^{-\frac{1}{4}-3\epsilon}$ and $|\mathcal{W}_{k, \mathbf{\Omega}'}| \leq CB_w p_k$ imply as in part I that $|\delta V_k(\square, \phi, \mathcal{W}_{k, \mathbf{\Omega}'})| \leq \lambda_k^{\frac{1}{4}-10\epsilon}$. (We had a sharper bound on W_k there, but the argument stills holds.)

Furthermore if $|t| \leq \lambda_k^{-\frac{1}{4}}$ we have $|t\mathcal{W}_{k, \mathbf{\Omega}'}| \leq CB_w p_k \lambda_k^{-\frac{1}{4}} \leq \frac{1}{2}\lambda_k^{-\frac{1}{4}-3\epsilon}$. There are similar bounds on derivatives and so $t\mathcal{W}_{k, \mathbf{\Omega}'} \in \frac{1}{2}\mathcal{R}_k$. Thus $t \rightarrow E_k(X, \phi_{k+1, \mathbf{\Omega}'}^0 + t\mathcal{W}_{k, \mathbf{\Omega}'})$ is analytic in $|t| \leq \lambda_k^{-1/4}$ we have the representation

$$\delta E_k(X, \phi, \mathcal{W}_{k, \mathbf{\Omega}'}) = \frac{1}{2\pi i} \int_{|t|=\lambda_k^{-1/4}} \frac{dt}{t(t-1)} E_k(X, \phi + t\mathcal{W}_{k, \mathbf{\Omega}'}) \quad (406)$$

Now $|E_k(X, \phi)| \leq \lambda_k^\beta e^{-\kappa d_M(X)}$ for $\phi \in \mathcal{R}_k$ is our basic assumption, and hence $|\delta E_k(X, \phi, \mathcal{W}_{k, \mathbf{\Omega}'})| \leq \mathcal{O}(1)\lambda_k^{\frac{1}{4}+\beta} e^{-\kappa d_M(X)}$. Altogether then

$$|\delta E_k^+(X, \phi, \mathcal{W}_{k, \mathbf{\Omega}'})| \leq \mathcal{O}(1)\lambda_k^{\frac{1}{4}-10\epsilon} e^{-\kappa d_M(X)} \quad (407)$$

We also reblock to get an element of $\mathcal{D}_{k+1, \mathbf{\Omega}'}^0$; since $X \subset \Lambda_k \subset \Omega_k$ this means preserving the M cubes in $\delta\Omega_k$ and replacing M cubes by LM cubes in Ω_{k+1} . We define for $Y \in \mathcal{D}_{k+1, \mathbf{\Omega}'}^0$

$$(\delta E_k^+)'(Y) = \sum_{X: \bar{X}=Y, X \subset \Lambda_k} (\delta E_k^+)'(X) \quad (408)$$

where for $X \in \mathcal{D}_k$, \bar{X} is the smallest element of $\mathcal{D}_{k+1, \mathbf{\Omega}'}^0$ containing X . We postpone the estimate on this quantity. Then we have $\delta E_k^+(\Lambda_k) = \sum_Y (\delta E_k^+)'(Y)$.

(B.) In $\mathcal{W}_{k, \mathbf{\Omega}'}$ we have the propagator $G_{k, \mathbf{\Omega}(\Lambda_k^*)}$. This has a random walk expansion based on the cubes of $\mathbf{\Omega}(\Lambda_k^*)$. It is convenient to use a modification in which we use the cubes of $\mathbf{\Omega}' = (\mathbf{\Omega}(\Lambda_k^*), \Omega_{k+1})$ instead, i.e. we take LM -cubes in Ω_{k+1} rather than M cubes. This gives a new random walk expansion, but it leads to exactly the same bounds as the expansion of theorem 2.2. This is true since the basic estimate on $G_{k, \mathbf{\Omega}(\Lambda_k^*)}(\tilde{\square})$ of lemma 2.6 holds for LM -cubes just as well as M -cubes, and since the fact that a cube can have $\mathcal{O}(L^2)$ neighbors is already built into the proof of theorem 2.2.

Then we can introduce the propagator $G_{k, \mathbf{\Omega}(\Lambda_k^*)}(s)$ with parameters $s = \{s_\square\}$ which weaken the coupling through multiscale cubes \square in $\mathbf{\Omega}'$. We also use again the weakened operator $C_{k, \mathbf{\Omega}'}^{1/2}(s)$ defined

in section 3.8. Replace $\mathcal{W}_{k,\mathbf{\Omega}'}$ by $\mathcal{W}_{k,\mathbf{\Omega}'}(s) = a_k G_{k,\mathbf{\Omega}(\Lambda_k^*)}(s) Q_k^T (C_{k,\mathbf{\Omega}'}^{1/2}(s) W_k)$. This does not spoil any of our estimates, even if we allow s_\square to be complex and satisfy $|s_\square| \leq M^{\frac{1}{2}} \equiv e^{\kappa_1}$. Then we have a decoupling expansion

$$(\delta E_k^+)'(Y, \phi, \mathcal{W}_{k,\mathbf{\Omega}'}) = \sum_{Y_1 \supset Y} (\delta E_k^+)'(Y, Y_1, \phi, W_k) \quad (409)$$

where $Y_1 \in \mathcal{D}_{k+1,\mathbf{\Omega}'}^0$ is a multiscale polymer and

$$(\delta E_k^+)'(Y, Y_1, \phi, W_k) = \int ds_{Y_1-Y} \frac{\partial}{\partial s_{Y_1-Y}} \left[(\delta E_k^+)'(Y, \phi, \mathcal{W}_{k,\mathbf{\Omega}'}(s)) \right]_{s_{Y^c}=0, s_Y=1} \quad (410)$$

The function $(\delta E_k^+)'(Y, Y_1, \phi, W_k)$ depends on $\mathbf{\Omega}', \Lambda_k$, has fields strictly localized in Y_1 , and vanishes unless $Y_1 \subset \Omega_1(\Lambda_k^*) \subset \Omega_k$. As in part I we use Cauchy bounds to estimate the derivatives $\partial/\partial s_{Y_1-Y}$ for $|s_\square| \leq 1$. This gains a factor $e^{-(\kappa_1-1)}$ for each $\square \subset Y_1 - Y$ and gives an overall improvement of our bounds, still postponed, by a factor of $\exp(-(\kappa_1-1)|Y_1 - Y|_{\mathbf{\Omega}'})$.

Now for $Y_1 \in \mathcal{D}_{k+1,\mathbf{\Omega}'}^0$ define the function $(\delta E_k^+)''(Y_1) = (\delta E_k^+)''(Y_1, \phi, W_k)$ by

$$(\delta E_k^+)''(Y_1) = \sum_{Y \subset Y_1} (\delta E_k^+)'(Y, Y_1) \quad (411)$$

This depends on $\mathbf{\Omega}', \Lambda_k$ and we have

$$\delta E_k^+(\Lambda_k) = \sum_{Y_1} (\delta E_k^+)''(Y_1) \quad (412)$$

(C.) Consider terms in (412) with $Y_1 \subset \Lambda_{k+1}$ and hence $Y_1 \in \mathcal{D}_{k+1}^0$. In this case in the expression (410) we are evaluating $\mathcal{W}_{k,\mathbf{\Omega}'}(s)$ with $s_{\Lambda_{k+1}^c} = 0$. In the random walk expansions defining this object only paths in Λ_{k+1} contribute, and these are the same for $G_{k,\mathbf{\Omega}(\Lambda_k^*)}(s), C_{k,\mathbf{\Omega}'}^{1/2}(s)$ and the global $G_k(s), C_k^{1/2}$ with LM cubes. Hence in this circumstance $\mathcal{W}_{k,\mathbf{\Omega}'}(s)$ is the same as the global $\mathcal{W}_k(s)$ and hence $\delta E_k^+(Y, Y_1)$ is the same as the global function of part I. Then $(\delta E_k^+)''(Y_1)$ is independent of $\mathbf{\Omega}', \Lambda_k$ and is equal to the global function $(\delta E_k^+)^{loc}(Y_1)$ of part I.

For $Y_1 \subset \Lambda_{k+1}$ we have $|Y_1 - Y|_{\mathbf{\Omega}'} = |Y_1 - Y|_{LM}$ and as in part I this leads to the estimate

$$|(\delta E_k^+)^{loc}(Y_1)| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-2\kappa_0-2)d_{LM}(Y_1)} \quad (413)$$

The sum of these terms in (412) is the desired expression

$$\sum_{Y_1 \in \mathcal{D}_{k+1}^0, Y_1 \subset \Lambda_{k+1}} (\delta E_k^+)^{loc}(Y_1) \quad (414)$$

(D.) Now consider terms in (412) with $Y_1 \cap \Lambda_{k+1}^c \neq \emptyset$. Weaken the coupling in $(\delta E_k^+)''(Y_1; \phi_{k+1,\mathbf{\Omega}'}^0, W_k)$ by replacing $\phi_{k+1,\mathbf{\Omega}'}^0$ by $\phi_{k+1,\mathbf{\Omega}'}^0(s)$ where again $s = s_\square$ is indexed by elementary cubes in $\mathbf{\Omega}'$. Then we have a second decoupling expansion

$$(\delta E_k^+)''(Y_1; \phi_{k+1,\mathbf{\Omega}'}^0, W_k) = \sum_{Y_2 \supset Y_1} (\delta E_k^+)''(Y_1, Y_2; \Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}}, W_k) \quad (415)$$

where for $Y_2 \in \mathcal{D}_{k+1}^0, \Omega'$

$$\begin{aligned} & (\delta E_k^+)''(Y_1, Y_2; \Phi_k, \delta\Omega_k, \Phi_{k+1}, \Omega_{k+1}, W_k) \\ &= \int ds_{Y_2-Y_1} \frac{\partial}{\partial s_{Y_2-Y_1}} \left[(\delta E_k^+)''(Y_1, \phi_{k+1}^0, \Omega'(s), W_k) \right]_{s_{Y_2^c}=0, s_{Y_1}=1} \end{aligned} \quad (416)$$

The function $(\delta E_k^+)''(Y_1, Y_2)$ depends on the fields only in Y_2 . The derivatives improve our estimates by a factor $\exp(-(\kappa_1 - 1)|Y_2 - Y_1|_{\Omega'})$. If we now define

$$(\delta E_k^+)'''(Y_2) = \sum_{Y_1 \subset Y_2, Y_1 \cap \Lambda_{k+1}^c \neq \emptyset} (\delta E_k^+)''(Y_1, Y_2) \quad (417)$$

then our expression becomes $\sum_{Y_2} (\delta E_k^+)'''(Y_2)$

(E.) Next we pass from polymers $Y_2 \in \mathcal{D}_{k+1}^0, \Omega'$ to polymers $Z \in \mathcal{D}_{k+1}^0$. We define $(\delta E_k^+)^{(iv)}(Z) = (\delta E_k^+)^{(iv)}(Z, \Phi_k, \delta\Omega_k, \Phi_{k+1}, \Omega_{k+1}, W_k)$ by

$$(\delta E_k^+)^{(iv)}(Z) = \sum_{Y_2: \bar{Y}_2 = Z} (\delta E_k^+)'''(Y_2) \quad (418)$$

where now \bar{Y}_2 is the smallest element of \mathcal{D}_{k+1}^0 containing Y_2 . The function $(\delta E_k^+)^{(iv)}(Z)$ vanishes unless $Z \cap \Lambda_{k+1}^c \neq \emptyset, Z \cap \Lambda_k \neq \emptyset$ so we can write our expression as

$$\sum_{Z \in \mathcal{D}_{k+1}^0, Z \cap \Lambda_{k+1}^c \neq \emptyset, Z \cap \Lambda_k \neq \emptyset} (\delta E_k^+)^{(iv)}(Z) \quad (419)$$

(F.) We estimate $(\delta E_k^+)^{(iv)}(Z)$. Collecting all the contributions, and dropping conditions $X \subset \Lambda_k$ and $Y_1 \cap \Lambda_{k+1}^c \neq \emptyset$ and we have

$$\begin{aligned} |(\delta E_k^+)^{(iv)}(Z)| &\leq \mathcal{O}(1) \lambda_k^{\frac{1}{4}-10\epsilon} \sum_{Y_2: \bar{Y}_2 = Z} \sum_{Y_1 \subset Y_2} \sum_{Y \subset Y_1} \sum_{X: \bar{X} = Y} \\ &\exp \left(-(\kappa_1 - 1)|Y_2 - Y_1|_{\Omega'} - (\kappa_1 - 1)|Y_1 - Y|_{\Omega'} - \kappa d_M(X) \right) \end{aligned} \quad (420)$$

Now $Md_M(X) \geq LMd_{LM}(\bar{Y})$ since a tree joining the M cubes in X will also join the LM cubes in $\bar{Y} \in \mathcal{D}_{k+1}^0$. Thus we can extract a factor $\exp(-L(\kappa - \kappa_0)d_{LM}(\bar{Y}))$ leaving $\exp(-\kappa_0 d_M(X))$. Now

$$|Y_2 - Y_1|_{\Omega'} + |Y_1 - Y|_{\Omega'} = |Y_2 - Y|_{\Omega'} \geq |Y_2 - \bar{Y}|_{\Omega'} \geq |Z - \bar{Y}|_{LM} \quad (421)$$

The last step follows since in passing from $Y_2 - \bar{Y}$ to $Z - \bar{Y}$ we replace each elementary $\mathcal{D}_{k+1}^0, \Omega'$ cube in $Y_2 - \bar{Y}$ by the LM cube containing it, and this cannot increase the number of elementary cubes. Then we can extract a factor $\exp(-(\kappa_1/2 - 1)|Z - \bar{Y}|_{LM})$ which for M sufficiently large is less than $\exp(-L(\kappa - \kappa_0)|Z - \bar{Y}|_{LM})$. Now use the inequality from part I :

$$|Z - \bar{Y}|_{LM} + d_{LM}(\bar{Y}) \geq d_{LM}(Z) \quad (422)$$

to dominate the extracted factors by $\exp(-L(\kappa - \kappa_0)d_{LM}(Z))$. Thus we have

$$\begin{aligned} |(\delta E_k^+)^{(iv)}(Z)| &\leq \mathcal{O}(1) \lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa - \kappa_0)d_{LM}(Z)} \sum_{Y_2: \bar{Y}_2 = Z} \sum_{Y_1 \subset Y_2} \sum_{Y \subset Y_1} \sum_{X: \bar{X} = Y} \\ &\exp \left(-\frac{1}{2} \kappa_1 |Y_2 - Y|_{\Omega'} - \kappa_0 d_M(X) \right) \end{aligned} \quad (423)$$

For the sum over Y_1 we drop connectedness conditions and take

$$\sum_{Y \subset Y_1 \subset Y_2} 1 \leq \text{number of subsets of cubes in } Y_2 - Y_1 \leq 2^{|Y_2 - Y_1|} \Omega' \quad (424)$$

This is absorbed by replacing $\frac{1}{2}\kappa_1$ by $\frac{1}{4}\kappa_1$ in (423). Now use $\sum_{Y_2 \subset Z} \sum_{Y \subset Y_2} \leq \sum_{Y \subset Z} \sum_{Y_2 \supset Y}$. Then the sum over Y_2 is estimated by lemma D.2 in the appendix (here Y, Y_2 are connected and we use LM cubes)

$$\sum_{Y_2 \supset Y} e^{-\frac{1}{4}\kappa_1|Y_2 - Y|} \Omega' \leq \exp\left(Ce^{-\frac{1}{8}\kappa_1|Y|} \Omega'\right) \leq \exp\left(Ce^{-\frac{1}{8}\kappa_1|Z|_{LM}}\right) \leq \mathcal{O}(1)e^{d_{LM}(Z)} \quad (425)$$

The second inequality holds since $|Y|_{\Omega'} \leq L^3|Z|_{LM}$. In the last step we have used the inequality $|Z|_{LM} \leq \mathcal{O}(1)(1 + d_{LM}(Z))$ and suppressed the constants by taking M and hence κ_1 large enough. In the final sum $\sum_{Y \subset Z} \sum_{X: \bar{X}=Y} = \sum_{X \subset Z}$ and

$$\sum_{X \subset Z} e^{-\kappa_0 d_M(X)} \leq \mathcal{O}(1)|Z|_M = \mathcal{O}(1)L^3|Z|_{LM} \leq \mathcal{O}(1)L^3(d_{LM}(Z) + 1) \leq \mathcal{O}(1)L^3e^{d_{LM}(Z)} \quad (426)$$

Then taking $L(\kappa - \kappa_0) - 2 \geq L(\kappa - \kappa_0 - 2)$ yields

$$|(\delta E_k^+)^{(iv)}(Z)| \leq \mathcal{O}(1)L^3\lambda_k^{\frac{1}{4}-10\epsilon}e^{-L(\kappa-\kappa_0-2)d_{LM}(Z)} \quad (427)$$

(G.) Terms in (419) with $Z \subset \Lambda_{k+1}^c$ are the $\tilde{B}_{k+1, \mathbf{\Pi}^+}$ terms in (403). These are estimated by

$$\begin{aligned} & \mathcal{O}(1)L^3\lambda_k^{\frac{1}{4}-10\epsilon} \sum_{Z: Z \subset \Lambda_{k+1}^c, Z \cap \Lambda_k \neq \emptyset} e^{-L(\kappa-\kappa_0-2)d_{LM}(Z)} \\ & \leq \mathcal{O}(1)L^3\lambda_k^{\frac{1}{4}-10\epsilon} \sum_{\square \subset \bar{\Lambda}_k - \Lambda_{k+1}} \sum_{Z \supset \square} e^{-L(\kappa-\kappa_0-2)d_{LM}(Z)} \leq \mathcal{O}(1)L^3\lambda_k^{\frac{1}{4}-10\epsilon} |\bar{\Lambda}_k - \Lambda_{k+1}|_{LM} \end{aligned} \quad (428)$$

where the sum is over LM cubes \square . Since $|\bar{\Lambda}_k - \Lambda_{k+1}|_{LM} = M^{-3}|\bar{\Lambda}_k^{(k+1)} - \Lambda_{k+1}^{(k+1)}|$ this is a bound of the required form.

(H.) The remaining terms in (419) satisfy $Z \# \Lambda_{k+1}$ and yield the active boundary terms $B_{k, \mathbf{\Pi}^+}^{(E)}$ terms in (403). First note that each such Z determines a $Z^+ \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c)$ by taking the union with all connected components of Ω_{k+1}^c are not disjoint from Z , written $Z \rightarrow Z^+$. We define

$$B_{k, \mathbf{\Pi}^+}^{(E)}(Y) = \sum_{Z \# \Lambda_{k+1}, Z^+ = Y} (\delta E_k^+)^{(iv)}(Z) \quad (429)$$

Then we have

$$\sum_{Z \# \Lambda_{k+1}} (\delta E_k^+)^{(iv)}(Z) = \sum_{Y \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c), Y \# \Lambda_{k+1}} B_{k, \mathbf{\Pi}^+}^{(E)}(Y) \quad (430)$$

To estimate $B_{k, \mathbf{\Pi}^+}^{(E)}(Y)$ first note that

$$d_{LM}(Z) \geq d_{LM}(Z^+, \text{mod } \Omega_{k+1}^c) \quad (431)$$

Indeed let τ be a minimal tree joining the cubes in Z of length $\ell(\tau) = LMd_{LM}(Z)$. Then τ is also a tree joining the cubes in $Z^+ \cap \Omega_{k+1}$ since $Z^+ \cap \Omega_{k+1} = Z \cap \Omega_{k+1} \subset Z$. Hence $\ell(\tau) \geq LMd_{LM}(Z^+, \text{mod } \Omega_{k+1}^c)$ and hence the result. Using this and (427) gives

$$|B_{k, \mathbf{\Pi}^+}^{(E)}(Y)| \leq \mathcal{O}(1)L^3\lambda_k^{\frac{1}{4}-10\epsilon} e^{-(L(\kappa-\kappa_0-2)-\kappa_0)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \sum_{Z \subset Y, Z \# \Lambda_{k+1}} e^{-\kappa_0 d_{LM}(Z)} \quad (432)$$

But the sum is bounded by

$$\begin{aligned}\mathcal{O}(1)|Y \cap \Lambda_{k+1}|_{LM} &\leq |Y \cap \Omega_{k+1}|_{LM} \leq \mathcal{O}(1)(d_{LM}(Y \cap \Omega_{k+1}) + 1) \\ &= \mathcal{O}(1)(d_{LM}(Y, \text{mod } \Omega_{k+1}^c) + 1) \leq \mathcal{O}(1)e^{d_{LM}(Y, \text{mod } \Omega_{k+1}^c)}\end{aligned}\quad (433)$$

The coefficient of $d_{LM}(Y \cap \Omega_{k+1})$ is then $L(\kappa - \kappa_0 - 2) - \kappa_0 - 1 \geq L(\kappa - 2\kappa_0 - 3)$ and we have the result.

Lemma 3.16. *The function $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(\leq 7)}$ can be written*

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(\leq 7)} = \sum_{Y \in \mathcal{D}_{k+1}^0: Y \subset \Lambda_{k+1}} R_{k, \mathbf{\Pi}^+}^{\text{loc}}(Y) + \sum_{Y \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c), Y \# \Lambda_{k+1}} B_{k, \mathbf{\Pi}^+}^{(R)}(Y) + \tilde{B}_{k+1, \mathbf{\Pi}^+} \text{ terms} \quad (434)$$

Here $R_{k, \mathbf{\Pi}^+}^{\text{loc}}(Y, \Phi_{k+1}, W_k)$ and $B_{k, \mathbf{\Pi}^+}^{(R)}(Y, \Phi_{k, \delta \Omega_k}, \Phi_{k+1, \Omega_{k+1}}, W_k)$ are strictly localized in the fields. They are analytic for Φ_k, Φ_{k+1} in the domain (376) and $|W_k| \leq B_w p_k$. On this domain they satisfy

$$\begin{aligned}|R_{k, \mathbf{\Pi}^+}^{\text{loc}}(Y)| &\leq \mathcal{O}(1)L^3 \lambda_k^{n_0} e^{-L(\kappa - \kappa_0 - 2)d_{LM}(Y)} \\ |B_{k, \mathbf{\Pi}^+}^{(R)}(Y)| &\leq \mathcal{O}(1)L^3 \lambda_k^{n_0} e^{-L(\kappa - 2\kappa_0 - 3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)}\end{aligned}\quad (435)$$

Proof. The function $R_{\mathbf{\Pi}^+, \Omega_{k+1}}^{(\leq 7)}(\Lambda_k)$ has many parts, which we consider one by one.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(0)} = R_{k, \mathbf{\Pi}}(\Lambda_k)$. This original term after the change of variables has the form

$$R_{k, \mathbf{\Pi}}(\Lambda_k) = \sum_{X \in \mathcal{D}_k, X \subset \Lambda_k} R_{k, \mathbf{\Pi}}(X, \Phi_{k, \delta \Omega_k}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k) \quad (436)$$

Our hypotheses on the fields and lemma 3.14 imply that $(\Phi_{k, \delta \Omega_k}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}')) \in \frac{1}{2}\mathcal{P}_k(\Lambda_k, 2\delta)$. We argue below that $|W_k| \leq B_w p_k$ implies that $(\Phi_{k, \delta \Omega_k}, (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k) \in \frac{1}{2}\mathcal{P}_k(\Lambda_k, 2\delta)$. Then the sum is in $\mathcal{P}_k(\Lambda_k, 2\delta)$, hence we are in the analyticity domain for $R_{k, \mathbf{\Pi}}(X)$, and hence

$$|R_{k, \mathbf{\Pi}}(X)| \leq \lambda_k^{n_0} e^{-\kappa d_M(X)} \quad (437)$$

For the missing piece let $W'_k = (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$. By the bound (305) $|W'_k| \leq C p_k$. Then for $\square \subset \Omega_{k+1}^\natural$ we have $|W'_k - \phi_{k, \mathbf{\Omega}'}(\square)(W'_k)|$, $|\partial \phi_{k, \mathbf{\Omega}'}(\square)(W'_k)|$, and $|\phi_{k, \mathbf{\Omega}'}(\square)(W'_k)|$ all bounded by $C p_k$. Since $C p_k \leq \frac{1}{2} \lambda_k^{-\frac{1}{4} - 2\delta}$ this gives $W'_k \in \frac{1}{2}\mathcal{P}_k(\square, 2\delta)$. If $\square \subset \Lambda_k - \Omega_{k+1}^\natural$ argue as in parts (B.), (C.) of lemma 3.14 to obtain $(\Phi_{k, \delta \Omega_k}, W'_k) \in \frac{1}{2}\mathcal{P}_k(\square, 2\delta)$. Altogether then we have $(\Phi_{k, \delta \Omega_k}, W'_k) \in \frac{1}{2}\mathcal{P}_k(\Lambda_k, 2\delta)$.

Now reblock as in (408) defining for $Y \in \mathcal{D}_{k+1, \mathbf{\Omega}'}^0$

$$R'_{k, \mathbf{\Pi}}(Y) = \sum_{X: \bar{X}=Y, X \subset \Lambda_k} R_{k, \mathbf{\Pi}}(X) \quad (438)$$

Then $R_{k, \mathbf{\Pi}}(\Lambda_k) = \sum_Y R'_{k, \mathbf{\Pi}}(Y)$.

We localize further as follows. Again introduce parameters $s = \{s_\square\}$ indexed by the cubes of $\mathbf{\Omega}'$. Referring to the definitions in section 3.7, for an LM cube \square in Ω_{k+1} we replace $G_{k+1, \mathbf{\Omega}'}^0(\square^*) =$

$G_{k+1,\mathbf{\Omega}'}^0(s_{\square^*} = 1, s_{\square^{*,c}} = 0)$ with normal coupling in \square^* by $G_{k+1,\mathbf{\Omega}'}^0(\square^*, s) = G_{k+1,\mathbf{\Omega}'}^0(s_{\square^*}, s_{\square^{*,c}} = 0)$ with weakened coupling inside \square^* . Correspondingly we replace $\phi_{k+1,\mathbf{\Omega}'}^0(\square^*)$ by $\phi_{k+1,\mathbf{\Omega}'}^0(\square^*, s)$, we replace $\Psi_{k,\Omega_{k+1}}(\mathbf{\Omega}', \square^*)$ by $\Psi_{k,\Omega_{k+1}}(\mathbf{\Omega}', \square^*, s)$, and we replace $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}')$ by $\Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}', s)$. Also referring to the definitions in section 3.8, we replace $G_{k,\mathbf{\Omega}',r}(\square^*)$ by $G_{k,\mathbf{\Omega}',r}(\square^*, s)$ with weakened coupling inside \square^* , we replace $C_{k,\mathbf{\Omega}',r}(\square^*)$ by $C_{k,\mathbf{\Omega}',r}(\square^*, s)$, we replace $C_{k,\mathbf{\Omega}'}^{1/2}(\square^*)$ by $C_{k,\mathbf{\Omega}'}^{1/2}(\square^*, s)$, and we replace $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}$ by $(C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}(s)$. None of these changes affect the bounds on the fields, even for $|s_{\square}| \leq e^{\kappa_1}$. Finally instead of $R'_{k,\mathbf{\Pi}}(Y, \Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k)$ we introduce

$$R'_{k,\mathbf{\Pi}}(Y, s) = R_{k,\mathbf{\Pi}}\left(Y, \Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}', s) + (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}(s)W_k\right) \quad (439)$$

Now make a decoupling expansion as in the previous lemma. We have

$$R'_{k,\mathbf{\Pi}}(Y, \Phi_{k,\delta\Omega_k}, \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}}W_k) = \sum_{Y_1 \supset Y} (R_{k,\mathbf{\Pi}^+})'(Y, Y_1, \Phi_{k,\delta\Omega_k}, \Phi_{k+1,\Omega_{k+1}}, W_k) \quad (440)$$

where for $Y_1 \in \mathcal{D}_{k+1,\mathbf{\Omega}'}$

$$(R_{k,\mathbf{\Pi}^+})'(Y, Y_1) = \int ds_{Y_1-Y} \frac{\partial}{\partial s_{Y_1-Y}} \left[R'_{k,\mathbf{\Pi}}(Y, s) \right]_{s_{Y_1^c}=0, s_Y=1} \quad (441)$$

depends on the indicated fields only in Y_1 . Note that in this case only terms with $Y_1 \subset \Omega_{k+1}^*$ contribute because of the sharper localization. Again using Cauchy bounds on the derivatives, we improve our estimates by a factor $\exp(-(\kappa_1 - 1)|Y_1 - Y|_{\mathbf{\Omega}'})$.

Next define

$$(R_{k,\mathbf{\Pi}^+})''(Y_1) = \sum_{Y \subset Y_1} (R_{k,\mathbf{\Pi}^+})'(Y, Y_1) \quad (442)$$

and then $R_{k,\mathbf{\Pi}}(\Lambda_k) = \sum_{Y_1} R_{k,\mathbf{\Pi}^+}''(Y_1)$. Further we reblock to $Z \in \mathcal{D}_{k+1}^0$ defining

$$(R_{k,\mathbf{\Pi}^+})'''(Z) = \sum_{Y_1: \bar{Y}_1=Z} (R_{k,\mathbf{\Pi}^+})''(Y_1) \quad (443)$$

and then

$$R_{k,\mathbf{\Pi}}(\Lambda_k) = \sum_{Z \cap \Lambda_k \neq \emptyset} R_{k,\mathbf{\Pi}^+}'''(Z) \quad (444)$$

Collecting our estimates we have

$$|R_{k,\mathbf{\Pi}^+}'''(Z)| \leq \mathcal{O}(1)\lambda_k^{n_0} \sum_{Y_1: \bar{Y}_1=Z} \sum_{Y \subset Y_1} \sum_{X: \bar{X}=Y} \exp\left(-(\kappa_1 - 1)|Y_1 - Y|_{\mathbf{\Omega}'} - \kappa d_M(X)\right) \quad (445)$$

This is estimated just as in part (F.) of the previous lemma, except that now there is just a sum over Y_1 instead of Y_1, Y_2 . The result is

$$|R_{k,\mathbf{\Pi}^+}'''(Z)| \leq \mathcal{O}(1)L^3\lambda_k^{n_0} e^{-L(\kappa - \kappa_0 - 2)d_{LM}(Z)} \quad (446)$$

Finally in (444) divide the terms into three classes as in the previous lemma. Terms with $Y \subset \Lambda_{k+1}$ contribute to $R_{k,\mathbf{\Pi}^+}^{\text{loc}}$. Terms with $Y \subset \Lambda_{k+1}^c$ are the $\tilde{B}_{k+1,\mathbf{\Pi}^+}$ terms, and have the correct bounds

as before. Terms with $Y \# \Lambda_{k+1}$ the boundary terms. We adjoin connected components of Ω_{k+1}^c as before, and get a contribution to $B_{k, \mathbf{\Pi}^+}^{(R)}(Y)$.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)}$. We have $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)} = \sum_{X \subset \Lambda_k} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)}(X)$ where for $X \in \mathcal{D}_k$

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)}(X) = E_k^+(X, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}} + \delta\phi_{k, \mathbf{\Omega}'} - E_k^+(X, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}) \quad (447)$$

Our field bounds imply $\phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}} \in \frac{1}{2}\mathcal{R}_k$. They also imply, arguing as in lemma 3.9, that $|\delta\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}')| \leq e^{-r_{k+1}}$ which yields the bound on Λ_k :

$$|\delta\phi_{k, \mathbf{\Omega}'}| = |\phi_{k, \mathbf{\Omega}(\Lambda_k^*)}(0, \delta\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}'))| \leq Ce^{-r_{k+1}} \quad (448)$$

with similar bounds on the derivatives. Hence for complex $|t| \leq e^{r_{k+1}}$ we have $t\delta\phi_{k, \mathbf{\Omega}'} \in \frac{1}{2}\mathcal{R}_k$. Now $t \rightarrow E_k^+(X, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}} + t\delta\phi_{k, \mathbf{\Omega}'})$ is analytic in $|t| \leq e^{r_{k+1}}$ and we can write

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)}(X) = \frac{1}{2\pi i} \int_{|t|=e^{r_{k+1}}} \frac{1}{t(t-1)} E_k^+(X, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}} + t\delta\phi_{k, \mathbf{\Omega}'}) \quad (449)$$

Using the bound $|E_k^+(X)| \leq \mathcal{O}(1)\lambda_k^{-12\epsilon}e^{-\kappa d_M(X)}$ on \mathcal{R}_k , we have (since $e^{-r_{k+1}} = \mathcal{O}(\lambda_k^n)$ for any n)

$$|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)}(X)| \leq \mathcal{O}(1)e^{-r_{k+1}}\lambda_k^{-12\epsilon}e^{-\kappa d_M(X)} \leq \lambda_k^{n_0}e^{-\kappa d_M(X)} \quad (450)$$

Now reblock as in (408) defining $(R^{(3)})'_{\mathbf{\Pi}, \Omega_{k+1}}(Y)$ for $Y \in \mathcal{D}_{k+1, \mathbf{\Omega}'}^0$. Next replace $\phi_{k+1, \mathbf{\Omega}'}^0$ by $\phi_{k+1, \mathbf{\Omega}'}^0(s)$ and $\mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}$ by $\mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}(s) \equiv a_k G_{k, \mathbf{\Omega}(\Lambda_k^*)}(s) Q_k^T((C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}}(s) W_k)$. Furthermore we replace $\delta\phi_{k, \mathbf{\Omega}'}$ by $\delta\phi_{k, \mathbf{\Omega}'}(s) \equiv a_k G_{k, \mathbf{\Omega}(\Lambda_k^*)}(s) Q_k^T \delta\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}', s)$ where $\delta\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}', s) = \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}', s) - \Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}', s)$. Bounds are unaffected and we get a weakened form $(R^{(3)})'_{\mathbf{\Pi}, \Omega_{k+1}}(Y, s)$ analogous to (439). Finally proceed with the decoupling and reblocking as in (440) - (446) and obtain contributions to $R_{k, \mathbf{\Pi}^+}^{\text{loc}}$, $\tilde{B}_{k+1, \mathbf{\Pi}^+}$, and $B_{k, \mathbf{\Pi}^+}^{(R)}(Y)$ of the required form.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(7)}$. This is entirely similar to $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(3)}$. We have $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(7)} = \sum_{X \subset \Lambda_k} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(7)}(X)$ where

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(7)}(X) = E_k^+(X, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'} + \delta\mathcal{W}_{k, \mathbf{\Omega}'} - E_k^+(X, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'}) \quad (451)$$

By the bound (305) on $\delta C_{k, \mathbf{\Omega}'}^{1/2}$ we have on Λ_k :

$$|\delta\mathcal{W}_{k, \mathbf{\Omega}'}| = a_k |G_{k, \mathbf{\Omega}(\Lambda_k^*)} Q_k^T(\delta C_{k, \mathbf{\Omega}'}^{1/2} W_k)| \leq Ce^{-r_{k+1}} p_k \quad (452)$$

So we can replace $\delta\mathcal{W}_{k, \mathbf{\Omega}'}$ by $t\delta\mathcal{W}_{k, \mathbf{\Omega}'}$ with $|t| \leq e^{r_{k+1}}$ and still stay in the region of analyticity. Hence we have the representation

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(7)}(X) = \frac{1}{2\pi i} \int_{|t|=e^{r_{k+1}}} \frac{dt}{t(t-1)} E_k^+(X, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}'} + t\delta\mathcal{W}_{k, \mathbf{\Omega}'}) \quad (453)$$

and the bound

$$|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(7)}(X)| \leq \mathcal{O}(1)e^{-r_{k+1}}\lambda_k^{-12\epsilon}e^{-\kappa d_M(X)} \leq \lambda_k^{n_0}e^{-\kappa d_M(X)} \quad (454)$$

Now reblock, decouple, and reblock again exactly as in the previous case, with the same result.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)}$. After the change of variables and a localization we have

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)} = \sum_{\substack{\square \text{ on } \partial \Lambda_k}} b_{\Lambda_k} \left[\partial \phi_{k+1, \mathbf{\Omega}'}^0, 1_{\square} \mathcal{W}_{k, \mathbf{\Omega}(\Lambda_k^*)}^{\text{loc}} \right] - \frac{1}{2} \sum_{\square \subset \Lambda_k^c} \left(a_k \|Q_{k, \mathbf{\Omega}(\Lambda_k^*)} \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{\square}^2 + \|\partial \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{*, \square}^2 + \bar{\mu}_k \|\mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{\square}^2 \right) \quad (455)$$

where the sum is over M -cubes \square . Then $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)} = \sum_{X \subset (\Lambda_k^c)^\sim} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)}(X)$ if we say $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)}(X)$ vanishes for $|X|_M \geq 2$.

Now $(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$ is localized in Ω_{k+1} which is separated from Λ_k^c by at least $5[r_{k+1}]$ layers of LM -cubes. Thus from the random walk expansion for $G_{k, \mathbf{\Omega}(\Lambda_k^*)}$ joining points in Λ_k^c and Ω_{k+1} we can extract a factor $e^{-r_{k+1}}$. Hence by a variation of (133) and have have on $(\Lambda_k^c)^\sim$:

$$|\mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}| = \left| \phi_{k, \mathbf{\Omega}(\Lambda_k^*)} \left((C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k \right) \right| \leq C e^{-r_{k+1}} \|(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k\|_{\infty} \leq C e^{-r_{k+1}} p_k \quad (456)$$

Near $\partial \Lambda_k$ we are in $\Omega_k(\Lambda_k^*)$ and as noted previously $|\partial \phi_{k+1, \mathbf{\Omega}'}^0| \leq C \lambda_k^{-\frac{1}{4}-2\delta}$. Also $|b_{\Lambda_k}[\partial \phi, 1_{\square} \mathcal{W}]| \leq M^2 \|\partial \phi\|_{\infty} \|\mathcal{W}\|_{\infty}$. These lead to the bound

$$b_{\Lambda_k} \left[\partial \phi_{k+1, \mathbf{\Omega}'}^0, 1_{\square} \mathcal{W}_{k, \mathbf{\Omega}(\Lambda_k^*)}^{\text{loc}} \right] \leq M^2 (C \lambda_k^{-\frac{1}{4}-2\delta}) (C e^{-r_{k+1}} p_k) \leq \lambda_k^{n_0} \quad (457)$$

Now consider the the term $\|\mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{\square}^2$ for $\square \subset \Lambda_k^c$. We have

$$\|\mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{\square}^2 \leq M^3 \|\mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{\infty}^2 \leq M^3 (C e^{-r_{k+1}} p_k)^2 \leq \lambda_k^{n_0} \quad (458)$$

The term $\|Q_{k, \mathbf{\Omega}(\Lambda_k^*)} \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{\square}^2$ is treated similarly. The derivative term needs more attention. For an $L^{-(k-j)}$ cube $\Delta_y \subset \delta \Omega_j(\Lambda_k^*) \cap \square$, by a variation of (138)

$$\begin{aligned} |1_{\Delta_y} \partial \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}| &= \left| \sum_{y' \in \Omega_{k+1}^{(k)}} 1_{\Delta_y} \partial \phi_{k, \mathbf{\Omega}(\Lambda_k^*)} \left(1_{\Delta_{y'}} (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k \right) \right| \\ &\leq \sum_{y' \in \Omega_{k+1}^{(k)}} C L^{k-j} e^{-r_{k+1}} e^{-\frac{1}{4} \gamma_0 d_{\mathbf{\Omega}(\Lambda_k^*)}(y, y')} \|(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k\|_{\infty} \\ &\leq C e^{-r_{k+1}} \|(C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k\|_{\infty} \leq C e^{-r_{k+1}} p_k \end{aligned} \quad (459)$$

Here we have used the fact that because of our separation conditions we have for $y \in \delta \Omega_j^{(j)}(\Lambda_k^*)$ and $y' \in \Omega_{k+1}^{(k)} \subset \Omega_k^{(k)}(\Lambda_k^*)$

$$d_{\mathbf{\Omega}(\Lambda_k^*)}(y, y') \geq RM \max\{|k-j|-1, 0\} \quad (460)$$

Then for M large enough the factor $\exp(-\frac{1}{4} \gamma_0 d_{\mathbf{\Omega}(\Lambda_k^*)}(y, y'))$ is enough to dominate the L^{k-j} and give convergence of the sum over y' . Now we have as before

$$\|\partial \mathcal{W}_{k, \mathbf{\Omega}'}^{\text{loc}}\|_{*, \square}^2 \leq M^3 (C e^{-r_{k+1}} p_k)^2 \leq \lambda_k^{n_0} \quad (461)$$

Altogether then

$$|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(1)}(\square)| \leq \mathcal{O}(1) \lambda_k^{n_0} \quad (462)$$

Now reblock, decouple, and reblock again as before, with the same result.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)}$. First we localize $J_{\Lambda_k, \Omega_{k+1}}^*$ defined in (87).

$$J_{\Lambda_k, \Omega_{k+1}}^*(\Phi_{k+1}, \Phi_k, \phi) = \sum_{\square \subset \Omega_{k+1}} \frac{a}{2L^2} \|\Phi_{k+1} - Q\Phi_k\|_{\square}^2 + \sum_{\square \subset \Lambda_k} S_k^*(\square, \Phi_k, \phi) \equiv \sum_{\square} J_{\Lambda_k, \Omega_{k+1}}^*(\square) \quad (463)$$

where the sum is over M -cubes \square . For $|\Phi_k|, |\Phi_{k+1}|, |\phi|, |\partial\phi| \leq C\lambda_k^{-\frac{1}{4}-2\delta}$ we have the bound

$$|J_{\Lambda_k, \Omega_{k+1}}^*(\square)| \leq CM^3 \lambda_k^{-\frac{1}{2}-4\delta} \quad (464)$$

Then $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)} = \sum_{\square} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)}(\square)$ where

$$\begin{aligned} & R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)}(\square) \\ &= J_{\Lambda_k, \Omega_{k+1}}^* \left(\square, \Phi_{k+1}, \hat{\Psi}_{k, \mathbf{\Omega}'} + (C_{k, \mathbf{\Omega}'}^{1/2} W)^{\text{loc}} + \delta\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}'), \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}(\Lambda_k^*)}^{\text{loc}} + \delta\phi_{k, \mathbf{\Omega}'} \right) \\ & - J_{\Lambda_k, \Omega_{k+1}}^* \left(\square, \Phi_{k+1}, \hat{\Psi}_{k, \mathbf{\Omega}'} + (C_{k, \mathbf{\Omega}'}^{1/2} W)^{\text{loc}}, \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}(\Lambda_k^*)}^{\text{loc}} \right) \end{aligned} \quad (465)$$

The hypotheses of the lemma give the fields the $C\lambda_k^{-\frac{1}{4}-2\delta}$ bound as indicated in earlier steps, and so $|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)}(\square)| \leq CM^3 \lambda_k^{-\frac{1}{2}-4\delta}$. However we also know that $\delta\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}')$ and $\delta\phi_{k, \mathbf{\Omega}'}$ are $\mathcal{O}(e^{-r_{k+1}})$ so we can multiply these factors by complex $|t| \leq e^{r_{k+1}}$ and still have the same bound. Therefore

$$\begin{aligned} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)}(\square) &= \frac{1}{2\pi i} \int_{|t|=e^{r_{k+1}}} \frac{dt}{t(t-1)} J_{\Lambda_k, \Omega_{k+1}}^* \left(\square, \Phi_{k+1}, \hat{\Psi}_{k, \mathbf{\Omega}'} + (C_{k, \mathbf{\Omega}'}^{1/2} W)^{\text{loc}} + t\delta\Psi_{k, \Omega_{k+1}}(\mathbf{\Omega}'), \right. \\ & \quad \left. \phi_{k+1, \mathbf{\Omega}'}^0 + \mathcal{W}_{k, \mathbf{\Omega}(\Lambda_k^*)}^{\text{loc}} + t\delta\phi_{k, \mathbf{\Omega}'} \right) dt \end{aligned} \quad (466)$$

which leads to the estimate

$$|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(2)}(\square)| \leq CM^3 \lambda_k^{-\frac{1}{2}-4\delta} e^{-r_{k+1}} \leq \lambda_k^{n_0} \quad (467)$$

Now reblock, decouple, and reblock again as before, with the same result.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(4)}$. First write $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(4)} = \sum_{\square \subset \Omega_{k+1}} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(4)}(\square)$ where the sum is over LM -cubes \square and

$$R_{\mathbf{\Pi}, \Omega_{k+1}}^{(4)}(\square) = \langle C_{k, \mathbf{\Omega}'}^{-1/2} W_k, 1_{\square} \delta C_{k, \mathbf{\Omega}'}^{1/2} W_k \rangle - \frac{1}{2} \langle \delta C_{k, \mathbf{\Omega}'}^{1/2} W_k, 1_{\square} C_{k, \mathbf{\Omega}'}^{-1} \delta C_{k, \mathbf{\Omega}'}^{1/2} W_k \rangle \quad (468)$$

Then taking bounds from lemma 3.6 and using $|W_k| \leq B_w p_k$ we have

$$\begin{aligned} |C_{k, \mathbf{\Omega}'}^{-1/2} W_k| &\leq C p_k \\ |\delta C_{k, \mathbf{\Omega}'}^{1/2} W_k| &\leq C p_k e^{-r_{k+1}} \\ |C_{k, \mathbf{\Omega}'}^{-1} \delta C_{k, \mathbf{\Omega}'}^{1/2} W_k| &\leq C p_k e^{-r_{k+1}} \end{aligned} \quad (469)$$

Therefore

$$|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(4)}(\square)| \leq M^3 C p_k e^{-r_{k+1}} \leq \lambda_k^{n_0} \quad (470)$$

Now reblock, decouple, and reblock again as before, with the same result.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(5)}$ The bounds of lemma 3.7 suffice. These terms contribute to $R_{k, \mathbf{\Pi}^+}^{\text{loc}}$ and $\tilde{B}_{k+1, \mathbf{\Pi}^+}$.

The term $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}$ Take the result $R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)} = \sum_{\square \subset \Omega_{k+1}} R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square)$ of lemma 3.8, and split into $\square \subset \Lambda_{k+1}$ and $\square \subset \Lambda_{k+1}^c$. In the former case $|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square)| \leq \lambda_k^{n_0}$ by (317) and the terms contribute to $R_{k, \mathbf{\Pi}^+}^{\text{loc}}$. In the latter case $|R_{\mathbf{\Pi}, \Omega_{k+1}}^{(6)}(\square)| \leq CM^3 = C \text{Vol}(\square)$ and the terms contribute to $\tilde{B}_{k+1, \mathbf{\Pi}^+}$. This completes the proof.

Lemma 3.17.

$$B_{k, \mathbf{\Pi}}(\Lambda_k) = \sum_{Y \in \mathcal{D}_{k+1}^0 \pmod{\Omega_{k+1}^c}, Y \# \Lambda_{k+1}} B_{k, \mathbf{\Pi}^+}^{(B)}(Y) + \tilde{B}_{k+1, \mathbf{\Pi}^+} \text{ terms} \quad (471)$$

where $B_{k, \mathbf{\Pi}^+}^{(B)}(Y, \Phi_{k+1, \mathbf{\Omega}^+}, W_{k+1, \mathbf{\Pi}^+}, W_{k, \Lambda_{k+1}})$ is strictly local in the fields. It is analytic in $\Phi_{k+1, \mathbf{\Omega}^+}$ in $\mathcal{P}_{k+1, \mathbf{\Omega}^+}^0$ and $|W_j| \leq B_w p_j L^{\frac{1}{2}(k-j)}$ for $j = 0, 1, \dots, k$, and satisfies there

$$|B_{k, \mathbf{\Pi}^+}^{(B)}(Y)| \leq \lambda_k^{n_0} e^{-L(\kappa - \kappa_0 - 3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \quad (472)$$

Remark. The proof is similar to the proof for the first term in lemma 3.16, except that now there are holes which localize spectator variables left over from the early stages of the expansion.

Proof. We are studying

$$B_{k, \mathbf{\Pi}}(\Lambda_k) = \sum_{X \in \mathcal{D}_k \pmod{\Omega_k^c}, X \# \Lambda_k} B_{k, \mathbf{\Pi}}(X, \Phi_{k, \mathbf{\Omega} \cap \Omega_{k+1}^c}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k, W_{k, \mathbf{\Pi}}) \quad (473)$$

Our assumptions on the fields and lemma (3.14) imply that $(\Phi_{k, \mathbf{\Omega} \cap \Omega_{k+1}^c}, \Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k) \in \mathcal{P}_{k, \mathbf{\Omega}}$. Together with the bounds on W_j this shows that we are in the domain of analyticity for $B_{k, \mathbf{\Pi}}(X)$ and have

$$|B_{k, \mathbf{\Pi}}(X)| \leq B_0 \lambda_k^\beta e^{-\kappa d_M(X, \text{mod } \Omega_k^c)} \quad (474)$$

If it happens that $X \subset \Omega_{k+1}^c$, then there is no dependence on the nonlocal fields $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$. Such terms sum up to a contribution to $\tilde{B}_{k+1, \mathbf{\Pi}^+}$.

For the remaining terms we reblock to polymers $Y \in \mathcal{D}_{k+1}^0$ by

$$B'_{k, \mathbf{\Pi}}(Y) = \sum_{\bar{X} = Y : X \cap \Omega_{k+1}^c \neq \emptyset, X \# \Lambda_k} B_{k, \mathbf{\Pi}}(X) \quad (475)$$

which depends on the same variables.

The only non-locality in $B'_{k, \mathbf{\Pi}}(Y)$ comes from the $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$ in $Y \cap \Omega_{k+1}$. Hence we temporarily treat $B'_{k, \mathbf{\Pi}}(Y)$ as localized in $Y \cap \Omega_{k+1}$. We introduce weakening parameters $\{s_\square\}$ for elementary cubes \square in $\mathbf{\Omega}'$, replace $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k$ by $\Psi_{k, \Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}', s) + (C_{k, \mathbf{\Omega}'}^{1/2})^{\text{loc}}(s) W_k$ and call the result $B'_{k, \mathbf{\Pi}}(Y, s)$. As in (475) this is a sum over certain $B_{k, \mathbf{\Pi}}(X, s)$ which satisfy (474).

Now we make the decoupling expansion based on $Y \cap \Omega_{k+1}$. It is a little different from the previous expansions since $Y \cap \Omega_{k+1}$ is not necessarily connected. We have

$$\begin{aligned} B'_{k,\mathbf{\Pi}}(Y, \Phi_{k,\Omega \cap \Omega_{k+1}^c}, \Psi_{k,\Omega_{k+1}}^{\text{loc}}(\mathbf{\Omega}') + (C_{k,\mathbf{\Omega}'}^{1/2})^{\text{loc}} W_k) \\ = \sum_{Y_1 \supset (Y \cap \Omega_{k+1})} B_{k,\mathbf{\Pi}^+}(Y, Y_1, \Phi_{k+1,\mathbf{\Omega}^+}, W_{k+1,\mathbf{\Pi}^+}, W_{k,\Lambda_{k+1}}) \end{aligned} \quad (476)$$

Here Y_1 is a multiscale object for $\mathbf{\Omega}'$ which may not be connected, but has the property that each connected component of Y_1 contains at least one connected component of Y . We define

$$(B_{k,\mathbf{\Pi}^+})'(Y, Y_1) = \int ds_{Y_1 - (Y \cap \Omega_{k+1})} \frac{\partial}{\partial s_{Y_1 - (Y \cap \Omega_{k+1})}} \left[B'_{k,\mathbf{\Pi}}(Y, s) \right]_{s_{Y_1^c} = 0, s_{Y \cap \Omega_{k+1}} = 1} \quad (477)$$

which depends on the indicated fields only in $Y \cup Y_1$. We have made the identifications $\Phi_{k+1,\mathbf{\Omega}^+} = (\Phi_{k,\Omega \cap \Omega_{k+1}^c}, \Phi_{k+1,\Omega_{k+1}})$ and $W_{k+1,\mathbf{\Pi}^+} = (W_{k,\mathbf{\Pi}}, W_{k,\Omega_{k+1} - \Lambda_{k+1}})$. Once again only terms with $Y_1 \subset \Omega_{k+1}^*$ contribute here. Using Cauchy bounds we improve our estimate on $B'_{k,\mathbf{\Pi}}(Y, s)$ by a factor $\exp(-(\kappa_1 - 1)|Y_1 - (Y \cap \Omega_{k+1})|_{\mathbf{\Omega}'})$.

Let \bar{Y}_1 be all LM cubes intersecting Y_1 and let $Z_0 = Y \cup \bar{Y}_1$. This is connected and hence an element of \mathcal{D}_{k+1}^0 . Then let $Z = Z_0^+$ be the union of Z_0 with any connected components of Ω_{k+1}^c not disjoint with Z_0 . Then $Z \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c)$ and the composite process is denoted $Y, Y_1 \rightarrow Z$. We define for such Z

$$(B_{k,\mathbf{\Pi}^+})''(Z) = \sum_{Y, Y_1 \rightarrow Z, Y_1 \supset (Y \cap \Omega_{k+1})} (B_{k,\mathbf{\Pi}^+})'(Y, Y_1) \quad (478)$$

and then

$$B_{k,\mathbf{\Pi}}(\Lambda_k) = \sum_{Z \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c), Z \# \Lambda_k, Z \cap \Omega_{k+1} \neq \emptyset} (B_{k,\mathbf{\Pi}^+})''(Z) + \tilde{B}_{k+1,\mathbf{\Pi}^+} \text{ terms} \quad (479)$$

Collecting our estimates we have

$$\begin{aligned} |(B_{k,\mathbf{\Pi}^+})''(Z)| &\leq \mathcal{O}(1) B_0 \lambda_k^\beta \\ &\sum_{Y, Y_1 \rightarrow Z, Y_1 \supset Y \cap \Omega_{k+1}} e^{-(\kappa_1 - 1)|Y_1 - (Y \cap \Omega_{k+1})|_{\mathbf{\Omega}'}} \sum_{\bar{X} = Y, X \cap \Omega_{k+1} \neq \emptyset, X \# \Lambda_k} e^{-\kappa d_M(X, \text{mod } \Omega_k^c)} \end{aligned} \quad (480)$$

To extract a decay factor first note that for $\bar{X} = Y$

$$d_M(X, \text{mod } \Omega_k^c) \geq L d_{LM}(Y, \text{mod } \Omega_{k+1}^c) \quad (481)$$

This follows since if τ is a minimal tree on the M -cubes in $X \cap \Omega_k$ with $\ell(\tau) = M d_M(X, \text{mod } \Omega_k^c)$, then it is also a tree on the M -cubes in $X \cap \Omega_{k+1}$, and hence on the LM cubes in $Y \cap \Omega_{k+1}$. So $\ell(\tau) \geq L M d_{LM}(X, \text{mod } \Omega_{k+1}^c)$. We also note that

$$|Y_1 - (Y \cap \Omega_{k+1})|_{\mathbf{\Omega}'} \geq |\bar{Y}_1 - (Y \cap \Omega_{k+1})|_{LM} \quad (482)$$

and we can take this factor with a coefficient L by borrowing from the $\kappa_1 - 1$. Now we claim that

$$L M |\bar{Y}_1 - (Y \cap \Omega_{k+1})|_{LM} + L M d_{LM}(Y, \text{mod } \Omega_{k+1}^c) \geq L M d_{LM}(Z, \text{mod } \Omega_{k+1}^c) \quad (483)$$

To see this let τ be a minimal tree on the LM cubes in $Y \cap \Omega_{k+1}$ with $\ell(\tau) = L M d_M(X, \text{mod } \Omega_k)$. Let $\{\tau_\alpha\}$ be trees on the LM cubes on the connected components of $\bar{Y}_1 - (Y \cap \Omega_{k+1})$. Then τ joined

to the $\{\tau_\alpha\}$ gives a tree τ' with $\ell(\tau')$ equal to the right side of (483). See lemma 20 in part I for more details. The tree τ' is constructed to connect the LM cubes in

$$\begin{aligned} (Y \cap \Omega_{k+1}) \cup (\bar{Y}_1 - (Y \cap \Omega_{k+1})) &= (Y \cap \Omega_{k+1}) \cup \bar{Y}_1 \\ &\supset (Y \cup \bar{Y}_1) \cap \Omega_{k+1} = Z_0 \cap \Omega_{k+1} = Z \cap \Omega_{k+1} \end{aligned} \quad (484)$$

This shows that $\ell(\tau') \geq LMd_{LM}(Z, \text{mod } \Omega_{k+1})$, and hence (483) is established.

By the above remarks our estimate becomes

$$\begin{aligned} |(B_{k, \mathbf{\Pi}^+})''(Z)| &\leq \mathcal{O}(1) \lambda_k^\beta B_0 e^{-L(\kappa - \kappa_0)d_{LM}(Z, \text{mod } \Omega_{k+1}^c)} \\ &\sum_{Y, Y_1 \rightarrow Z, Y_1 \supset Y \cap \Omega_{k+1}} e^{-\kappa_1/2|Y_1 - (Y \cap \Omega_{k+1})|} |\mathbf{\Omega}'| \sum_{\bar{X}=Y, X \cap \Omega_{k+1} \neq \emptyset} e^{-\kappa_0 d_M(X, \text{mod } \Omega_k^c)} \end{aligned} \quad (485)$$

Relax the sum over Y, Y_1 to just $Y \subset Z, Y_1 \supset Y \cap \Omega_{k+1}$. The sum over Y_1 is estimated by lemma D.2 by

$$\sum_{Y_1 \supset Y \cap \Omega_{k+1}} e^{-\kappa_1/2|Y_1 - (Y \cap \Omega_{k+1})|} |\mathbf{\Omega}'| \leq e^{Ce^{-\kappa_1/4}|Y \cap \Omega_{k+1}|_{LM}} \leq \mathcal{O}(1) e^{d_{LM}(Z, \text{mod } \Omega_{k+1}^c)} \quad (486)$$

The second step follows by $|Y \cap \Omega_{k+1}|_{LM} \leq |Z \cap \Omega_{k+1}|_{LM} \leq \mathcal{O}(1)(d_{LM}(Z, \text{mod } \Omega_{k+1}^c) + 1)$ as in (433). The constants are suppressed by taking M and hence κ_1 sufficiently large. Identifying $\sum_{Y \subset Z} \sum_{\bar{X}=Y}$ as $\sum_{X \subset Z}$ and using (151) the remaining sum is dominated by

$$\begin{aligned} \sum_{X \subset Z, X \cap \Omega_{k+1} \neq \emptyset} e^{-\kappa_0 d_M(X, \text{mod } \Omega_k^c)} &\leq \mathcal{O}(1) |Z \cap \Omega_{k+1}|_M \\ &= \mathcal{O}(1) L^3 |Z \cap \Omega_{k+1}|_{LM} \leq \mathcal{O}(1) L^3 e^{d_{LM}(Z, \text{mod } \Omega_{k+1}^c)} \end{aligned} \quad (487)$$

Thus we obtain for $Z \# \Lambda_k$ and $Z \cap \Omega_{k+1} \neq \emptyset$

$$|(B_{k, \mathbf{\Pi}^+})''(Z)| \leq \mathcal{O}(1) L^3 B_0 \lambda_k^\beta e^{-L(\kappa - \kappa_0 - 2)d_{LM}(Z, \text{mod } \Omega_{k+1}^c)} \quad (488)$$

In the sum (479) consider terms with $Z \subset \Lambda_{k+1}^c$. These are $\tilde{B}_{k+1, \mathbf{\Pi}^+}$ terms for they have the proper localization, and the sum of these terms can be estimated by $\mathcal{O}(1) L^3 B_0 \lambda_k^\beta$ times

$$\begin{aligned} \sum_{Z \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c), Z \# \Lambda_k, Z \subset \Lambda_{k+1}^c} e^{-L(\kappa - \kappa_0 - 2)d_{LM}(Z, \text{mod } \Omega_{k+1}^c)} \\ \leq \mathcal{O}(1) |\bar{\Lambda}_k - \Lambda_{k+1}|_{LM} \leq \mathcal{O}(1) |\bar{\Lambda}_k^{(k+1)} - \Lambda_{k+1}^{(k+1)}| \end{aligned} \quad (489)$$

Now consider terms in (479) with $Z \# \Lambda_{k+1}$. These are the terms $(B_{k, \mathbf{\Pi}^+}^{(B)})(Z) = (B_{k, \mathbf{\Pi}^+})''(Z)$. Since also $Z \# \Lambda_k$ we have $Z \# \Omega_{k+1}$ and so Z must have cubes in Ω_{k+1} on the boundary and in Λ_{k+1} , and these are necessarily a distance at least $r_{k+1}LM$ apart. Then any tree joining the LM cubes in $Z \cap \Omega_{k+1}$ must have length at least $r_{k+1}LM$. Hence $LMd_{LM}(Z, \text{mod } \Omega_{k+1}^c) \geq LMr_{k+1}$ and therefore $d_{LM}(Z, \text{mod } \Omega_{k+1}^c) \geq r_{k+1}$. We use this to extract a tiny factor $e^{-r_{k+1}}$ leaving say $e^{-L(\kappa - \kappa_0 - 3)d_M(Z, \text{mod } \Omega_k^c)}$.

Then for λ_k sufficiently small take $\mathcal{O}(1) L^3 B_0 \lambda_k^\beta e^{-r_{k+1}} \leq \lambda_k^{n_0}$. This is the basic mechanism which keeps the boundary terms from growing. Terms which survive many steps must have a large extent and hence a tiny value. Altogether then we have the announced bound

$$|B_{k, \mathbf{\Pi}^+}^{(B)}(Z)| \leq \lambda_k^{n_0} e^{-L(\kappa - \kappa_0 - 3)d_{LM}(Z, \text{mod } \Omega_{k+1}^c)} \quad (490)$$

This completes the proof.

Summary: We collect the boundary terms by

$$B_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y) = B_{k,\mathbf{\Pi}^+}^{(E)}(Y) + B_{k,\mathbf{\Pi}^+}^{(R)}(Y) + B_{k,\mathbf{\Pi}^+}^{(B)}(Y) \quad (491)$$

Inserting the results of the last three lemmas we have for the fluctuation integral:

$$\begin{aligned} \Xi'_{k,\mathbf{\Pi}^+} &= \exp\left(\tilde{B}_{k+1} \text{ terms}\right) \Xi''_{k,\mathbf{\Pi}^+} \\ \Xi''_{k,\mathbf{\Pi}^+} &= \int d\mu_{\Lambda_{k+1}}^*(W_k) \exp\left(\sum_{Y \subset \Lambda_{k+1}} (\delta E_k^+)^{\text{loc}}(Y) + R_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y) + \sum_{Y \# \Lambda_{k+1}} B_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y)\right) \end{aligned} \quad (492)$$

Here $Y \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c)$ and $(\delta E_k^+)^{\text{loc}}(Y) = (\delta E_k^+)^{\text{loc}}(Y, \phi, W_k)$ at $\phi = \phi_{k+1,\mathbf{\Omega}^+}^0$. It is analytic in $\phi \in \frac{1}{2}\mathcal{R}_k$ and $|W_k| \leq B_w p_k$ and (relaxing the bounds a bit) satisfies there

$$|(\delta E_k^+)^{\text{loc}}(Y)| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y)} \quad (493)$$

Also $R_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y, \Phi_{k+1}, W_k)$ is analytic in the domain (376) and $|W_k| \leq B_w p_k$ and satisfies there

$$|R_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y)| \leq \mathcal{O}(1) L^3 \lambda_k^{n_0} e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y)} \quad (494)$$

Also $B_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y, \Phi_{k+1,\mathbf{\Omega}^+}, W_{k+1,\mathbf{\Pi}^+}, W_{k,\Lambda_{k+1}})$ is analytic in the domain $\Phi_{k+1,\mathbf{\Omega}^+} \in \mathcal{P}_{k+1}^0, \mathbf{\Omega}^+$ and $|W_j| \leq B_w p_j L^{\frac{1}{2}(k-j)}$ and satisfies there

$$|B_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y)| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \quad (495)$$

We can also write

$$\Xi''_{k,\mathbf{\Pi}^+} = \int d\mu_{\Lambda_{k+1}}^*(W_k) \exp\left(\sum_Y H_{k,\mathbf{\Pi}^+}(Y)\right) \quad (496)$$

where the sum is over $Y \in \mathcal{D}_{k+1}^0(\text{mod } \Omega_{k+1}^c)$ and

$$H_{k,\mathbf{\Pi}^+}(Y) = \left((\delta E_k^+)^{\text{loc}}(Y) + R_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y)\right) 1_{Y \subset \Lambda_{k+1}} + B_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y) 1_{Y \# \Lambda_{k+1}} \quad (497)$$

This is analytic in the smallest of the three domains which is

$$\Phi_{k+1,\mathbf{\Omega}^+} \in \mathcal{P}_{k+1,\mathbf{\Omega}^+}^0 \quad |W_j| \leq B_w p_j L^{\frac{1}{2}(k-j)} \quad (498)$$

Furthermore if $Y \subset \Lambda_{k+1} \subset \Omega_{k+1}$ then $d_{LM}(Y) = d_{LM}(Y, \text{mod } \Omega_{k+1})$ and so

$$|H_{k,\mathbf{\Pi}^+}(Y)| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \quad (499)$$

3.14 cluster expansion

We want to perform a cluster expansion on the fluctuation integral $\Xi''_{k,\mathbf{\Pi}^+}$, and also isolate the most important terms which come from $(\delta E_k^+)^{\text{loc}}(Y)$. Accordingly we introduce some variables t, u which parametrize the contribution of the other terms and define

$$\Xi''_{k,\mathbf{\Pi}^+}(t, u) = \int d\mu_{\Lambda_{k+1}}^*(W_k) \exp\left(\sum_Y H_{k,\mathbf{\Pi}^+}(t, u, Y)\right) \quad (500)$$

where

$$H_{k,\mathbf{\Pi}^+}(t, u, Y) = \left((\delta E_k^+)^{\text{loc}}(Y) + t R_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y)\right) 1_{Y \subset \Lambda_{k+1}} + u B_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y) 1_{Y \# \Lambda_{k+1}} \quad (501)$$

We are interested in $\Xi''_{k,\mathbf{\Pi}^+}(1, 1) = \Xi''_{k,\mathbf{\Pi}^+}$, but start with a more general result:

Lemma 3.18. (*cluster expansion*) For $r_0 = \mathcal{O}(1)$ sufficiently small and $|t| \leq r_0 L^{-3} \lambda_k^{-n_0}$ and $|u| \leq r_0 L^{-3} \lambda^{-\frac{1}{4}+10\epsilon}$

$$\Xi''_{k,\mathbf{\Pi}^+}(t,u) = \exp \left(\sum_{Y \cap \Lambda_{k+1} \neq \emptyset} H_{k,\mathbf{\Pi}^+}^\#(t,u,Y) \right) \quad (502)$$

where $Y \in \mathcal{D}_{k+1}^0 \pmod{\Omega_{k+1}^c}$ and $H_{k,\mathbf{\Pi}^+}^\#(t,u,Y) = H_{k,\mathbf{\Pi}^+}^\#(t,u,Y,\phi,\Phi_{k+1,\mathbf{\Omega}^+},W_{k+1,\mathbf{\Pi}^+},)$ is evaluated at $\phi = \phi_{k+1,\mathbf{\Omega}'}^0$. The function is analytic in t,u , and in $\phi \in \frac{1}{2}\mathcal{R}_k$ and $\Phi_{k+1,\mathbf{\Omega}^+}, W_{k+1,\mathbf{\Pi}^+}$ in (498) and on this domain it satisfies

$$|H_{k,\mathbf{\Pi}^+}^\#(t,u,Y)| \leq \mathcal{O}(1) e^{-L(\kappa-6\kappa_0-6)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \quad (503)$$

Proof. Since integrand is well-localized and the measure is ultralocal, we can use the cluster expansion with holes which can be found in appendix F, now with LM cubes. In the domain (498), which puts W_k in the support of $\mu_{\Lambda_{k+1}}^*$, we have

$$\begin{aligned} |u| |B_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y)| &\leq \mathcal{O}(1) r_0 e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \leq \frac{1}{3} c_0 e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \\ |t| |R_{k,\mathbf{\Pi}^+}^{\text{loc}}(Y)| &\leq \mathcal{O}(1) r_0 e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y)} \leq \frac{1}{3} c_0 e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \end{aligned} \quad (504)$$

and $|(\delta E_k^+)^{\text{loc}}(Y)|$ satisfies the same bound for λ_k sufficiently small. Thus altogether

$$|H_{k,\mathbf{\Pi}^+}(t,u,Y)| \leq c_0 e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \quad (505)$$

This is the input for the cluster expansion. The output is the representation (502) with a bound which implies (503).

Lemma 3.19. (*removal of boundary terms*)

$$\Xi''_{k,\mathbf{\Pi}^+}(1,1) = \exp \left(\sum_{Y \# \Lambda_{k+1}} B_{k,\mathbf{\Pi}^+}^\#(Y) \right) \Xi_{k,\mathbf{\Pi}^+}(1,0) \quad (506)$$

with $B_{k,\mathbf{\Pi}^+}^\#(Y) = B_{k,\mathbf{\Pi}^+}^\#(Y,\phi,\Phi_{k+1,\mathbf{\Omega}^+},W_{k+1,\mathbf{\Pi}^+})$ evaluated at $\phi = \phi_{k+1,\mathbf{\Omega}'}^0$. It is analytic in $\phi \in \frac{1}{2}\mathcal{R}_k$ and (498) and satisfies there

$$|B_{k,\mathbf{\Pi}^+}^\#(Y)| \leq \mathcal{O}(1) L^3 \lambda^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-6\kappa_0-6)d_{LM}(Y, \text{mod } \Omega_{k+1}^c)} \quad (507)$$

Proof. We have

$$\Xi''_{k,\mathbf{\Pi}^+}(1,1) = \exp \left(\sum_{Y \cap \Lambda_{k+1} \neq \emptyset} (H_{k,\mathbf{\Pi}^+}^\#(1,1,Y) - H_{k,\mathbf{\Pi}^+}^\#(1,0,Y)) \right) \Xi_{k,\mathbf{\Pi}^+}(1,0) \quad (508)$$

so the identity holds with $B_{k,\mathbf{\Pi}^+}^\#(Y) = H_{k,\mathbf{\Pi}^+}^\#(1,1,Y) - H_{k,\mathbf{\Pi}^+}^\#(1,0,Y)$ or

$$B_{k,\mathbf{\Pi}^+}^\#(Y) = \frac{1}{2\pi i} \int_{|u|=r_0 L^{-3} \lambda_k^{-\frac{1}{4}+10\epsilon}} \frac{du}{u(u-1)} H_{k,\mathbf{\Pi}^+}^\#(1,u,Y) \quad (509)$$

Note that if $Y \subset \Lambda_{k+1}$ then no boundary term $B_{k,\mathbf{\Pi}^+}^{loc}(Y)$ can contribute. This is a consequence of the local influence property of the cluster expansion. Hence $H_{k,\mathbf{\Pi}^+}^\#(1, u, Y)$ is independent of u and so $B_{k,\mathbf{\Pi}^+}^\#(Y) = 0$. Therefore the sum in (508) is actually over $Y \# \Lambda_{k+1}$ as claimed. The bound (503) on $H_{k,\mathbf{\Pi}^+}^\#(1, u, Y)$ for $|u| = r_0 L^{-3} \lambda_k^{-\frac{1}{4} + 10\epsilon}$ now implies that $B_{k,\mathbf{\Pi}^+}^\#(Y)$ satisfies (507). This completes the proof.

Lemma 3.20. *(removal of tiny terms)*

$$\Xi''_{k,\mathbf{\Pi}^+}(1, 0) = \exp \left(\sum_{Y \subset \Lambda_{k+1}} R_{k,\mathbf{\Pi}^+}^\#(Y) \right) \Xi''_{k,\mathbf{\Pi}^+}(0, 0) \quad (510)$$

with $R_{k,\mathbf{\Pi}^+}^\#(Y) = R_{k,\mathbf{\Pi}^+}^\#(Y, \phi, \Phi_{k+1})$ evaluated at $\phi = \phi_{k+1}^0, \mathbf{\Omega}'$. It is analytic in $\phi \in \frac{1}{2}\mathcal{R}_k$ and $\Phi_{k+1} \in \mathcal{P}_{k+1}^0(\Lambda_{k+1}, 2\delta)$ and satisfies there

$$|R_{k,\mathbf{\Pi}^+}^\#(Y)| \leq \mathcal{O}(1) L^3 \lambda_k^{n_0} e^{-L(\kappa - 6\kappa_0 - 6)d_{LM}(Y)} \quad (511)$$

Proof. Now we are back to the standard cluster expansion with no holes and all polymers contained in Λ_{k+1} . $H_{k,\mathbf{\Pi}^+}(t, 0, Y)$ is analytic in $|t| \leq r_0 L^{-3} \lambda_k^{-n_0}$ with the bound $|H_{k,\mathbf{\Pi}^+}(t, 0, Y)| \leq c_0 e^{-L(\kappa - 3\kappa_0 - 3)d_{LM}(Y)}$. Therefore $H_{k,\mathbf{\Pi}^+}^\#(t, 0, Y)$ is analytic in the same domain with bound

$$|H_{k,\mathbf{\Pi}^+}^\#(t, 0, Y)| \leq \mathcal{O}(1) e^{-L(\kappa - 6\kappa_0 - 6)d_{LM}(Y)} \quad (512)$$

Now we have

$$\Xi''_{k,\mathbf{\Pi}^+}(1, 0) = \exp \left(\sum_{Y \subset \Lambda_{k+1}} (H_{k,\mathbf{\Pi}^+}^\#(1, 0, Y) - H_{k,\mathbf{\Pi}^+}^\#(0, 0, Y)) \right) \Xi''_{k,\mathbf{\Pi}^+}(0, 0) \quad (513)$$

so the identity holds with

$$R_{k,\mathbf{\Pi}^+}^\#(Y) = H_{k,\mathbf{\Pi}^+}^\#(1, 0, Y) - H_{k,\mathbf{\Pi}^+}^\#(0, 0, Y) = \frac{1}{2\pi i} \int_{|t|=r_0 L^{-3} \lambda_k^{-n_0}} \frac{dt}{t(t-1)} H_{k,\mathbf{\Pi}^+}^\#(t, 0, Y) \quad (514)$$

The bound (512) now implies that $R_{k,\mathbf{\Pi}^+}^\#(Y)$ satisfies (511). This completes the proof.

Now we are reduced to $\Xi''_{k,\mathbf{\Pi}^+}(0, 0)$ which is

$$\Xi''_{k,\mathbf{\Pi}^+}(0, 0) = \int d\mu_{\Lambda_{k+1}}^*(W_k) \exp \left(\sum_{Y \subset \Lambda_{k+1}} (\delta E_k^+)^{loc}(Y, \phi, W_k) \right) \quad (515)$$

at $\phi = \phi_{k+1}^0, \mathbf{\Omega}'$. Before the evaluation in ϕ this is just the quantity considered in part I, except that the sum over Y is restricted to Λ_{k+1} and the measure is restricted to Λ_{k+1} .

Lemma 3.21. *(leading terms)*

$$\Xi''_{k,\mathbf{\Pi}^+}(0, 0) = \exp \left(\sum_{Y \subset \Lambda_{k+1}} E_k^\#(Y, \phi) \right) \quad \text{at} \quad \phi = \phi_{k+1}^0, \mathbf{\Omega}' \quad (516)$$

where $E_k^\#(Y, \phi)$ is analytic in $\frac{1}{2}\mathcal{R}_k$ and satisfies there

$$|E_k^\#(Y, \phi)| \leq \mathcal{O}(1) L^3 \lambda_k^{\frac{1}{4} - 10\epsilon} e^{-L(\kappa - 6\kappa_0 - 6)d_{LM}(Y)} \quad (517)$$

$E_k^\#(Y, \phi)$ is identical with the function constructed in the global small field analysis in part I.

Proof. This is again the standard cluster expansion, taking account that $(\delta E_k^+)^{\text{loc}}(Y)$ is analytic in $\frac{1}{2}\mathcal{R}_k$ and has the bound $\mathcal{O}(1)L^3\lambda_k^{\frac{1}{4}-10\epsilon}e^{-L(\kappa-3\kappa_0-3)d_{LM}(Y)}$. The function $E_k^\#(Y, \phi)$ defined here is the same as the global definition in part I, even though here we are only summing over polymers in Λ_{k+1} . This is so since by the local influence property and the fact that the $(\delta E_k^+)^{\text{loc}}(Y)$ are the same. This completes the proof.

Combining the above results yields

$$\Xi''_{k, \mathbf{\Pi}^+} = \exp \left(E_k^\#(\Lambda_{k+1}) + R_{k, \mathbf{\Pi}^+}^\#(\Lambda_{k+1}) + B_{k, \mathbf{\Pi}^+}^\#(\Lambda_{k+1}) \right) \quad (518)$$

where $E_k^\#(\Lambda_{k+1}) = E_k^\#(\Lambda_{k+1}, \phi_{k+1, \mathbf{\Omega}'})$. Inserting this back into (492) and (401) yields

$$\begin{aligned} \Xi_{k, \mathbf{\Pi}^+} = \exp \Big(& -\varepsilon_k^0 \text{Vol}(\Lambda_{k+1}) + E_k^+(\Lambda_k) \\ & + E_k^\#(\Lambda_{k+1}) + R_{k, \mathbf{\Pi}^+}^\#(\Lambda_{k+1}) + B_{k, \mathbf{\Pi}^+}^\#(\Lambda_{k+1}) + \tilde{B}_{k+1, \mathbf{\Pi}^+} \text{ terms} \Big) \end{aligned} \quad (519)$$

Finally inserting this back in (398)

$$\begin{aligned} \tilde{\rho}_{k+1}(\Phi_{k+1}) = & Z_{k+1}^0 \sum_{\mathbf{\Pi}^+} \int d\Phi_{k+1, \mathbf{\Omega}^{+,c}}^0 dW_{k+1, \mathbf{\Pi}^+}^0 K_{k, \mathbf{\Pi}} C_{k+1, \mathbf{\Pi}^+}^0 \exp \left(c_{k+1} |\Omega_{k+1}^{c, (k)}| \right) \\ & \chi_{k+1}^0(\Lambda_{k+1}) \exp \left(-S_{k+1}^{*,0}(\Lambda_k) - \varepsilon_k^0 \text{Vol}(\Lambda_{k+1}) + E_k^+(\Lambda_k) \right. \\ & \left. + E_k^\#(\Lambda_{k+1}) + R_{k, \mathbf{\Pi}^+}^\#(\Lambda_{k+1}) + B_{k, \mathbf{\Pi}^+}^\#(\Lambda_{k+1}) + \tilde{B}_{k+1, \mathbf{\Pi}^+} \text{ terms} \right) \end{aligned} \quad (520)$$

3.15 scaling

We scale and evaluate $\rho_{k+1}(\Phi_{k+1}) = \tilde{\rho}(\Phi_{k+1, L})L^{-|\mathbb{T}_{M+N-k}^1|/2}$ where now Φ_{k+1} is defined on $\mathbb{T}_{M+N-(k+1)}^0$. We make the following changes in this expression. This follows the discussion in section 2.2.

- Identify $Z_{k+1} = Z_{k+1}^0 L^{-|\mathbb{T}_{M+N-k}^1|/2}$
- The sum over regions $\mathbf{\Omega}^+ = (\Omega_1, \dots, \Omega_{k+1})$ with Ω_j a union of $L^{-(k-j)}M$ blocks in \mathbb{T}_{M+N-k}^{-k} is relabeled as $L\mathbf{\Omega}^+ = (L\Omega_1, \dots, L\Omega_{k+1})$ where now $\mathbf{\Omega}^+ = (\Omega_1, \dots, \Omega_{k+1})$ with Ω_j a union of $L^{-(k+1-j)}M$ blocks in $\mathbb{T}_{M+N-k-1}^{-k-1}$. Similarly the sum over $\mathbf{\Lambda}^+$ is replaced by a sum over $L\mathbf{\Lambda}^+$ and the sum over $\mathbf{\Pi}^+$ is replaced by a sum over $L\mathbf{\Pi}^+$.
- The fields $\Phi_{\mathbf{\Omega}^+} = (\Phi_{1, \delta\Omega_1}, \dots, \Phi_{k+1, \delta\Omega_{k+1}})$ defined on subsets of \mathbb{T}_{M+N-k}^{-k} which has become $\Phi_{L\mathbf{\Omega}^+} = (\Phi_{1, L\delta\Omega_1}, \dots, \Phi_{k+1, L\delta\Omega_{k+1}})$. Now make a change of variables replacing $\Phi_{j, L\delta\Omega_j}$ by $[\Phi_{j, L}]_{L\delta\Omega_j} = [\Phi_{j, \delta\Omega_j}]_L$. Then $\Phi_{L\mathbf{\Omega}^+}$ becomes $\Phi_{\mathbf{\Omega}^+, L} = (\Phi_{1, \delta\Omega_{1, L}}, \dots, \Phi_{k+1, \delta\Omega_{k+1, L}})$. Furthermore the measure $d\Phi_{k+1, \mathbf{\Omega}^{+,c}}^0$ becomes $d\Phi_{k+1, \mathbf{\Omega}^{+,c}}$.
- Similarly we make a change of variables in W replacing $W_{j, L\Omega_j - L\Lambda_j}$ by $[W_{j, \Omega_j - \Lambda_j}]_L$. Then $dW_{k+1, \mathbf{\Pi}^+}^0$ becomes $dW_{k+1, \mathbf{\Pi}^+}$.
- Under these changes $\chi_{k+1}^0(\Lambda_{k+1})$ becomes $\chi_{k+1}(\Lambda_{k+1})$. Also if we define

$$\mathcal{C}_{k+1, \Lambda_k, \Omega_{k+1}, \Lambda_{k+1}}(\Phi_k, W_k, \Phi_{k+1}) = \mathcal{C}_{k+1, L\Lambda_k, L\Omega_{k+1}, L\Lambda_{k+1}}(\Phi_{k, L}, W_{k, L}, \Phi_{k+1, L}) \quad (521)$$

then $\mathcal{C}_{k+1, \mathbf{\Pi}^+}^0$ becomes $\mathcal{C}_{k+1, \mathbf{\Pi}^+}$ as defined in (201).

- We have already noted in (74) that $\phi_{k+1,\mathbf{\Omega}^+}^0$ becomes $\phi_{k+1,\mathbf{\Omega}^+,L}$. More to the point here $\phi_{k+1,\mathbf{\Omega}'}^0 = \phi_{k+1,\mathbf{\Omega}'}^0(\hat{\Phi}_{k+1,\mathbf{\Omega}'})$ becomes

$$\phi_{k+1,L\mathbf{\Omega}'}^0([\hat{\Phi}_{k+1,\mathbf{\Omega}'}]_L) = \phi_{k+1,\mathbf{\Omega}',L} \quad (522)$$

where now $\phi_{k+1,\mathbf{\Omega}'} = \phi_{k+1,\mathbf{\Omega}'}(\hat{\Phi}_{k+1,\mathbf{\Omega}'})$ with $\hat{\Phi}_{k+1,\mathbf{\Omega}'} = ([\tilde{Q}_{\mathbb{T}^{-1},\mathbf{\Omega}(\Lambda_k^*)}^T \Phi_k]_{\Omega_{k+1}^c}, \Phi_{k+1,\Omega_{k+1}})$.

- As noted earlier the action $S_{k+1}^{*,0}(\Lambda_k, \Phi_{k+1,\mathbf{\Omega}^+}, \phi_{k+1,\mathbf{\Omega}'}^0)$ becomes

$$S_{k+1}^{*,0}(L\Lambda_k, \Phi_{k+1,\mathbf{\Omega}^+,L}, \phi_{k+1,\mathbf{\Omega}',L}) = S_{k+1}^*(\Lambda_k, \Phi_{k+1,\mathbf{\Omega}^+}, \phi_{k+1,\mathbf{\Omega}'}) \quad (523)$$

We split this as $S_{k+1}^*(\Lambda_k) = S_{k+1}^*(\Lambda_k - \Lambda_{k+1}) + S_{k+1}^*(\Lambda_{k+1})$

- In $E_k^+(\Lambda_k) = E_k(\Lambda_k) - V_k(\Lambda_k)$ we have the potential $V_k(\Lambda_k, \phi_{k+1,\mathbf{\Omega}'}^0)$. This becomes

$$V_k(L\Lambda_k, \phi_{k+1,\mathbf{\Omega}',L}) = V_{k+1}^u(\Lambda_k, \phi_{k+1,\mathbf{\Omega}'}) \quad (524)$$

and we split this as $V_{k+1}^u(\Lambda_k) = V_{k+1}^u(\Lambda_k - \Lambda_{k+1}) + V_{k+1}^u(\Lambda_{k+1})$.

- Before scaling $E_k(\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}'}^0) = \sum_{X \in \mathcal{D}_k, X \subset \Lambda_{k+1}} E_k(X, \phi_{k+1,\mathbf{\Omega}'}^0)$ we apply a reblocking operation. For $Y \in \mathcal{D}_{k+1}^0$ define

$$(\mathcal{B}E)(Y) = \sum_{X \in \mathcal{D}_k, \bar{X}=Y} E_k(X) \quad (525)$$

where \bar{X} is the union of all LM cubes intersecting X . Then $E_k(\Lambda_k) = \sum_{Y \subset \Lambda_{k+1}} (\mathcal{B}E_k)(Y)$. Upon scaling this becomes (since $L\mathcal{D}_{k+1} = \mathcal{D}_{k+1}^0$)

$$\begin{aligned} E_k(L\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}',L}) &= \sum_{Y \in \mathcal{D}_{k+1}^0, Y \subset L\Lambda_{k+1}} (\mathcal{B}E_k)(Y, \phi_{k+1,\mathbf{\Omega}',L}) \\ &= \sum_{X \in \mathcal{D}_{k+1}, X \subset \Lambda_{k+1}} (\mathcal{B}E_k)(LX, \phi_{k+1,\mathbf{\Omega}',L}) \\ &= \sum_{X \in \mathcal{D}_{k+1}, X \subset \Lambda_{k+1}} (\mathcal{B}E_k)_{L^{-1}}(X, \phi_{k+1,\mathbf{\Omega}'}) \\ &\equiv (\mathcal{B}E_k)_{L^{-1}}(\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}'}) \end{aligned} \quad (526)$$

- The function $E_k^\#(\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}'}^0)$ is already reblocked, but otherwise is treated the same way. Under scaling it becomes

$$E_k^\#(L\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}',L}) \equiv (E_k^\#)_{L^{-1}}(\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}'}) \quad (527)$$

Similarly $R_{k,\mathbf{\Pi}^+}^\#(\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}'}^0, \Phi_{k+1})$ scales to

$$R_{k,L\mathbf{\Pi}^+}^\#(L\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}',L}, \Phi_{k+1,L}) \equiv [(R_k^\#)_{L^{-1}}]_{\mathbf{\Pi}^+}(\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}'}, \Phi_{k+1}) \quad (528)$$

- Now consider $B_{k,\mathbf{\Pi}^+}^\#(\Lambda_{k+1}, \phi_{k+1,\mathbf{\Omega}'}^0, \Phi_{k+1,\mathbf{\Omega}^+}, W_{k+1,\mathbf{\Pi}^+})$ which is also reblocked, and has the local decomposition

$$B_{k,\mathbf{\Pi}^+}^\#(\Lambda_{k+1}) = \sum_{Y \in \mathcal{D}_{k+1}^0 \pmod{\Omega_{k+1}^c}, Y \# \Lambda_{k+1}} B_{k,\mathbf{\Pi}^+}^\#(Y) \quad (529)$$

Upon scaling this becomes

$$\begin{aligned}
& B_{k,L\mathbf{\Pi}^+}^\#(L\Lambda_{k+1}, \phi_{k+1}, \mathbf{\Omega}'_{,L}, \Phi_{k+1}, \mathbf{\Omega}^+_{,L}, W_{k+1}, \mathbf{\Pi}^+_{,L}) \\
&= \sum_{Y \in \mathcal{D}_{k+1}^0(\text{mod } L\Omega_{k+1}^c), Y \# L\Lambda_{k+1}} (B_{k,L\mathbf{\Pi}^+}^\#)(Y, \phi_{k+1}, \mathbf{\Omega}'_{,L}, \Phi_{k+1}, \mathbf{\Omega}^+_{,L}, W_{k+1}, \mathbf{\Pi}^+_{,L}) \\
&= \sum_{X \in \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c), X \# \Lambda_{k+1}} (B_{k,L\mathbf{\Pi}^+}^\#)(LX, \phi_{k+1}, \mathbf{\Omega}'_{,L}, \Phi_{k+1}, \mathbf{\Omega}^+_{,L}, W_{k+1}, \mathbf{\Pi}^+_{,L}) \quad (530) \\
&= \sum_{X \in \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c), X \# \Lambda_{k+1}} [(B_k^\#)_{L^{-1}}]_{\mathbf{\Pi}^+}(X, \phi_{k+1}, \mathbf{\Omega}', \Phi_{k+1}, \mathbf{\Omega}^+, W_{k+1}, \mathbf{\Pi}^+) \\
&\equiv [(B_k^\#)_{L^{-1}}]_{\mathbf{\Pi}^+}(\Lambda_{k+1}, \phi_{k+1}, \mathbf{\Omega}', \Phi_{k+1}, \mathbf{\Omega}^+, W_{k+1}, \mathbf{\Pi}^+)
\end{aligned}$$

Here we have used that $L\mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c) = \mathcal{D}_{k+1}^0(\text{mod } L\Omega_{k+1}^c)$.

- Finally consider $K_{k,\mathbf{\Pi}}$ which scales to

$$\begin{aligned}
& [K_{k,L^{-1}}]_{\mathbf{\Pi}} \\
&\equiv \prod_{j=0}^k \exp \left(c_j |\Omega_j^{c,(j-1)}| - S_{j,L^{-(k+1-j)}}^{+,u} (\Lambda_{j-1} - \Lambda_j) + (\tilde{B}_{j,L^{-(k+1-j)}})_{\mathbf{\Pi}_j} (\Lambda_{j-1}, \Lambda_j) \right) \quad (531)
\end{aligned}$$

- Collect all the scaled " $\tilde{B}_{k+1,\mathbf{\Pi}^+}$ terms" into a single term $\tilde{B}_{k+1,\mathbf{\Pi}^+}(\Lambda_k, \Lambda_{k+1})$

With all these changes

$$\begin{aligned}
\rho_{k+1}(\Phi_{k+1}) &= Z_{k+1} \sum_{\mathbf{\Pi}^+} \int d\Phi_{\mathbf{\Omega}^+} dW_{\mathbf{\Pi}^+} [K_{k,L^{-1}}]_{\mathbf{\Pi}} \mathcal{C}_{k+1,\mathbf{\Pi}^+} \\
&\quad \exp \left(c_{k+1} |\Omega_{k+1}^{c,(k)}| - S_{k+1}^*(\Lambda_k - \Lambda_{k+1}) - V_{k+1}^u(\Lambda_k - \Lambda_{k+1}) + \tilde{B}_{k+1,\mathbf{\Pi}^+}(\Lambda_k, \Lambda_{k+1}) \right) \\
&\quad \chi_{k+1}(\Lambda_{k+1}) \exp \left(-S_{k+1}^*(\Lambda_{k+1}) - \varepsilon_k^0 L^3 \text{Vol}(\Lambda_{k+1}) - V_{k+1}^u(\Lambda_{k+1}) + (\mathcal{B}E_k)_{L^{-1}}(\Lambda_{k+1}) \right. \\
&\quad \left. + (E_k^\#)_{L^{-1}}(\Lambda_{k+1}) + [(R_k^\#)_{L^{-1}}]_{\mathbf{\Pi}^+}(\Lambda_{k+1}) + [(B_k^\#)_{L^{-1}}]_{\mathbf{\Pi}^+}(\Lambda_{k+1}) \right) \quad (532)
\end{aligned}$$

3.16 the RG flow

Now we show that the coupling constant flow follows the global analysis of part I, even though the effective action is localized. To do this we need to process the terms $(\mathcal{B}E_k)_{L^{-1}}(\Lambda_{k+1})$ and $(E_k^\#)_{L^{-1}}(\Lambda_{k+1})$.

First consider a more general case. Let $\Lambda \subset \mathbb{T}_{\mathbf{M}+\mathbf{N}-k-1}^{-k-1}$, and $\phi : \Lambda \rightarrow \mathbb{R}$, and suppose $E(\Lambda) = \sum_{X \subset \Lambda} E(X)$ with $E(X, \phi)$ translation invariant. Following the analysis in part I we make the following definitions. If X is small ($X \in \mathcal{S}$) then $\mathcal{R}E(X)$ is defined by

$$E(X, \phi) = \alpha_0(E, X) \text{Vol}(X) + \alpha_2(E, X) \int_X \phi^2 + \sum_{\mu} \alpha_{2,\mu}(E, X) \int_X \phi \partial_{\mu} \phi + \mathcal{R}E(X, \phi) \quad (533)$$

where

$$\begin{aligned}
\alpha_0(E, X) &= \frac{1}{\text{Vol}(X)} E(X, 0) & \alpha_2(E, X) &= \frac{1}{2 \text{Vol}(X)} E_2(X, 0; 1, 1) \\
\alpha_{2,\mu}(E, X) &= \frac{1}{\text{Vol}(X)} \left(E_2(X, 0; 1, x_{\mu} - x_{\mu}^0) - \frac{1}{\text{Vol}(X)} E_2(X, 0; 1, 1) \int_X x_{\mu} - x_{\mu}^0 \right) \quad (534)
\end{aligned}$$

The last is independent of the base point x^0 , which we take to be in X . With these choices $\mathcal{R}E(X)$ is normalized for small sets, that is the function and certain derivatives vanish at zero. If X is large then $\mathcal{R}E(X) = E(X)$.

Summing this over $X \subset \Lambda$ we find

$$\begin{aligned} E(\Lambda) = & - \sum_{\square \subset \Lambda} \varepsilon_\Lambda(E, \square) \text{Vol}(\square) - \frac{1}{2} \sum_{\square \subset \Lambda} \mu_\Lambda(E, \square) \|\phi\|_\square^2 \\ & - \sum_{\mu} \sum_{\square \subset \Lambda} \nu_{\Lambda, \mu}(E, \square) \int_{\square} \phi \partial_\mu \phi + \sum_{X \subset \Lambda} (\mathcal{R}E)(X) \end{aligned} \quad (535)$$

where

$$\begin{aligned} \varepsilon_\Lambda(E, \square) = & - \sum_{\square \subset X \subset \Lambda, X \in \mathcal{S}} \alpha(E, X) & \frac{1}{2} \mu_\Lambda(E, \square) = & - \sum_{\square \subset X \subset \Lambda, X \in \mathcal{S}} \alpha_2(E, X) \\ \nu_{\Lambda, \mu}(E, \square) = & - \sum_{\square \subset X \subset \Lambda, X \in \mathcal{S}} \alpha_{2, \mu}(E, X) \end{aligned} \quad (536)$$

Now if \square is well inside Λ then $X \in \mathcal{S}$ and $X \supset \square$ imply $X \subset \Lambda$ so we can drop the latter condition from the sums. Then $\varepsilon_\Lambda(E, \square)$, $\mu_\Lambda(E, \square)$, $\nu_{\Lambda, \mu}(E, \square)$ are independent of \square and Λ and agree with the global quantities which are denoted $\varepsilon(E)$, $\mu(E)$, $\nu_\mu(E)$. Furthermore the lattice symmetries imply that $\nu_\mu(E) = 0$. Then we write

$$\begin{aligned} E(\Lambda) = & - \varepsilon(E) \text{Vol}(\Lambda) - \frac{1}{2} \mu(E) \|\phi\|_\Lambda^2 + \sum_{X \subset \Lambda} (\mathcal{R}E)(X) - \sum_{\square \subset \Lambda} (\varepsilon_\Lambda(E, \square) - \varepsilon(E)) \text{Vol}(\square) \\ & - \frac{1}{2} \sum_{\square \subset \Lambda} (\mu_\Lambda(E, \square) - \mu(E)) \|\phi\|_\square^2 + \sum_{\mu} \sum_{\square \subset \Lambda} (\nu_{\Lambda, \mu}(E, \square) - \nu_\mu(E)) \int_{\square} \phi \partial_\mu \phi \end{aligned} \quad (537)$$

The last three terms can be treated as boundary terms. Indeed we have

$$\begin{aligned} \sum_{\square \subset \Lambda} (\varepsilon_\Lambda(E, \square) - \varepsilon(E)) \text{Vol}(\square) &= \sum_{\square \subset \Lambda} \left(\sum_{X \in \mathcal{S}, X \supset \square, X \not\subset \Lambda} \alpha_0(E, X) \right) \text{Vol}(\square) \\ &= \sum_{X \in \mathcal{S}, X \not\subset \Lambda} \alpha_0(E, X) \left(\sum_{\square \subset \Lambda \cap X} \text{Vol}(\square) \right) \\ &= \sum_{X \in \mathcal{S}, X \not\subset \Lambda} \alpha_0(E, X) \text{Vol}(\Lambda \cap X) \end{aligned} \quad (538)$$

Similarly

$$\frac{1}{2} \sum_{\square \subset \Lambda} (\mu_\Lambda(E, \square) - \mu(E)) \|\phi\|_\square^2 = \sum_{X \in \mathcal{S}, X \not\subset \Lambda} \alpha_{2,0}(X) \|\phi\|_{X \cap \Lambda}^2 \quad (539)$$

and

$$\sum_{\mu} \sum_{\square \subset \Lambda} (\nu_{\Lambda, \mu}(E, \square) - \nu_\mu(E)) \int_{\square} \phi \partial_\mu \phi = \sum_{X \in \mathcal{S}, X \not\subset \Lambda} \sum_{\mu} \alpha_{2, \mu}(E, X) \int_{X \cap \Lambda} \phi \partial_\mu \phi \quad (540)$$

we combine the last three terms defining for $X \in \mathcal{S}$ only

$$\mathcal{T}_\Lambda E(X, \phi) = \alpha_0(E, X) \text{Vol}(\Lambda \cap X) + \alpha_2(E, X) \|\phi\|_{X \cap \Lambda}^2 + \sum_{\mu} \alpha_{2, \mu}(E, X) \int_{X \cap \Lambda} \phi \partial_\mu \phi \quad (541)$$

Now (537) becomes

$$E(\Lambda) = -\varepsilon(E) \text{Vol}(\Lambda) - \frac{1}{2} \mu(E) \|\phi\|_\Lambda^2 + \sum_{X \subset \Lambda} (\mathcal{R}E)(X) + \sum_{X \not\subset \Lambda, X \in \mathcal{S}} \mathcal{T}_\Lambda E(X) \quad (542)$$

Now return to our specific problem. In the exponential in (532) we pick out the terms

$$\begin{aligned}
& -\varepsilon_k^0 L^3 \text{Vol}(\Lambda) - V_{k+1}^u(\Lambda) + (\mathcal{B}E_k)_{L^{-1}}(\Lambda) + (E_k^\#)_{L^{-1}}(\Lambda) \\
& = -(\varepsilon_k^0 + \varepsilon_k) L^3 \text{Vol}(\Lambda) - \frac{1}{2} L^2 \mu_k \|\phi\|_\Lambda^2 - \frac{1}{4} L \lambda_k \int_\Lambda \phi^4 + (\mathcal{B}E_k)_{L^{-1}}(\Lambda) + (E_k^\#)_{L^{-1}}(\Lambda)
\end{aligned} \tag{543}$$

evaluated at $\Lambda = \Lambda_{k+1}$ and $\phi = \phi_{k+1, \mathbf{\Omega}'}$. Applying (542) to the last two terms we have

$$\begin{aligned}
(\mathcal{B}E_k)_{L^{-1}}(\Lambda) & = -\varepsilon \left((\mathcal{B}E_k)_{L^{-1}} \right) \text{Vol}(\Lambda) - \frac{1}{2} \mu \left((\mathcal{B}E_k)_{L^{-1}} \right) \|\phi_k\|_\Lambda^2 \\
& + \sum_{X \subset \Lambda} \left(\mathcal{R}(\mathcal{B}E_k)_{L^{-1}} \right)(X) + \sum_{X \# \Lambda} \mathcal{T}_\Lambda(\mathcal{B}E_k)_{L^{-1}}(X) \\
& \equiv -\mathcal{L}_1(E_k) \text{Vol}(\Lambda) - \frac{1}{2} \mathcal{L}_2(E_k) \|\phi\|_\Lambda^2 + \sum_{X \subset \Lambda} (\mathcal{L}_3 E_k)(X) + \sum_{X \# \Lambda} \left(\mathcal{T}_\Lambda(\mathcal{B}E_k)_{L^{-1}} \right)(X)
\end{aligned} \tag{544}$$

$$\begin{aligned}
(E_k^\#)_{L^{-1}}(\Lambda) & = -\varepsilon \left((E_k^\#)_{L^{-1}} \right) \text{Vol}(\Lambda) - \frac{1}{2} \mu \left((E_k^\#)_{L^{-1}} \right) \|\phi_k\|_\Lambda^2 \\
& + \sum_{X \subset \Lambda} \left(\mathcal{R}(E_k^\#)_{L^{-1}} \right)(X) + \sum_{X \# \Lambda} \mathcal{T}_\Lambda(E_k^\#)_{L^{-1}}(X) \\
& \equiv -(\varepsilon_k^* - L^3 \varepsilon_k^0) \text{Vol}(\Lambda) - \frac{1}{2} \mu_k^* \|\phi\|_\Lambda^2 + \sum_{X \subset \Lambda} E_k^*(X) + \sum_{X \# \Lambda} \left(\mathcal{T}_\Lambda(E_k^\#)_{L^{-1}} \right)(X)
\end{aligned} \tag{545}$$

Insert these into (543) and identify the coupling constants at the next level. As in part I these are:

$$\begin{aligned}
\varepsilon_{k+1} & = L^3 \varepsilon_k + \mathcal{L}_1 E_k + \varepsilon_k^*(\lambda_k, \mu_k, E_k) \\
\mu_{k+1} & = L^2 \mu_k + \mathcal{L}_2 E_k + \mu_k^*(\lambda_k, \mu_k, E_k) \\
\lambda_{k+1} & = L \lambda_k \\
E_{k+1} & = \mathcal{L}_3 E_k + E_k^*(\lambda_k, \mu_k, E_k)
\end{aligned} \tag{546}$$

Now the terms (543) can be written:

$$\begin{aligned}
& -\varepsilon_{k+1} \text{Vol}(\Lambda) - \frac{1}{2} \mu_{k+1} \|\phi\|_\Lambda^2 - \frac{1}{4} \lambda_{k+1} \int_\Lambda \phi^4 + \sum_{X \subset \Lambda} E_{k+1}(X) + \sum_{X \# \Lambda} \left(\mathcal{T}_\Lambda(\mathcal{B}E_k + E_k^\#)_{L^{-1}} \right)(X) \\
& = -V_{k+1}(\Lambda) + E_{k+1}(\Lambda) + \left(\mathcal{T}_\Lambda(\mathcal{B}E_k + E_k^\#)_{L^{-1}} \right)(\Lambda)
\end{aligned} \tag{547}$$

still at $\Lambda = \Lambda_{k+1}$ and $\phi = \phi_{k+1, \mathbf{\Omega}'}$. We insert this back into (532).

3.17 more adjustments

We also make some changes in the fields. Currently we have the field $\phi_{k+1, \mathbf{\Omega}'}$. In the active terms we change this to the desired $\phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}$ defining tiny terms $R_{k+1, \mathbf{\Pi}^+}^{*,(i)}$ by

$$\begin{aligned}
S_{k+1}^*(\Lambda_{k+1}, \Phi_{k+1}, \phi_{k+1, \mathbf{\Omega}'}) & = S_{k+1}^*(\Lambda_{k+1}, \Phi_{k+1}, \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}) + R_{k+1, \mathbf{\Pi}^+}^{*,(1)} \\
V_{k+1}(\Lambda_{k+1}, \phi_{k+1, \mathbf{\Omega}'}) & = V_{k+1}(\Lambda_{k+1}, \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}) + R_{k+1, \mathbf{\Pi}^+}^{*,(2)} \\
E_{k+1}(\Lambda_{k+1}, \phi_{k+1, \mathbf{\Omega}'}) & = E_{k+1}(\Lambda_{k+1}, \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}) + R_{k+1, \mathbf{\Pi}^+}^{*,(3)}
\end{aligned} \tag{548}$$

In the inactive terms we change to the field $\phi_{k+1}, \Omega_{(\Lambda_k, \Omega_{k+1}, \Lambda_{k+1})}$ defining more tiny terms by

$$\begin{aligned} S_{k+1}^* \left(\delta \Lambda_k, \Phi_{k+1}, \Omega^+, \phi_{k+1}, \Omega' \right) &= S_{k+1}^* \left(\delta \Lambda_k, \Phi_{k+1}, \Omega^+, \phi_{k+1}, \Omega_{(\Lambda_k, \Omega_{k+1}, \Lambda_{k+1})} \right) + R_{k+1, \Pi^+}^{*,(4)} \\ V_{k+1}^u \left(\delta \Lambda_k, \phi_{k+1}, \Omega' \right) &= V_{k+1}^u \left(\delta \Lambda_k, \phi_{k+1}, \Omega_{(\Lambda_k, \Omega_{k+1}, \Lambda_{k+1})} \right) + R_{k+1, \Pi^+}^{*,(5)} \end{aligned} \quad (549)$$

where $\delta \Lambda_k = \Lambda_k - \Lambda_{k+1}$. With the new arguments we identify $S_{k+1}^+(\Lambda_{k+1}) = S_{k+1}^*(\Lambda_{k+1}) + V_{k+1}(\Lambda_{k+1})$, and $S_{k+1}^{+,u}(\delta \Lambda_k) = S_{k+1}^*(\delta \Lambda_k) + V_{k+1}^u(\delta \Lambda_k)$, and then

$$K_{k+1, \Pi^+} = [K_{k, L^{-1}}]_{\Pi^+} \exp \left(c_{k+1} |\Omega_{k+1}^{c, (k)}| - S_{k+1}^{+,u}(\delta \Lambda_k) + \tilde{B}_{k+1, \Pi^+}(\Lambda_k, \Lambda_{k+1}) \right) \quad (550)$$

Now collect the tiny and boundary terms defining $R_{k+1, \Pi^+}^{*,(0)} = [(R_k^\#)_{L^{-1}}]_{\Pi^+}(\Lambda_{k+1})$ and $B_{k+1, \Pi^+}^{*,(0)} = [(B_k^\#)_{L^{-1}}]_{\Pi^+}(\Lambda_{k+1})$ and $B_{k+1, \Pi^+}^{*,(1)} = (\mathcal{T}_{\Lambda_{k+1}}(\mathcal{B}E_k + E_k^\#)_{L^{-1}})(\Lambda_{k+1})$ and

$$\begin{aligned} R_{k+1, \Pi^+}^* &= R_{k+1, \Pi^+}^{*,(0)} + \dots + R_{k+1, \Pi^+}^{*,(5)} \\ B_{k+1, \Pi^+}^* &= B_{k+1, \Pi^+}^{*,(0)} + B_{k+1, \Pi^+}^{*,(1)} \end{aligned} \quad (551)$$

With these changes and (547) the representation (532) becomes

$$\begin{aligned} \rho_{k+1}(\Phi_{k+1}) &= Z_{k+1} \sum_{\Pi^+} \int d\Phi_{\Omega^+, c} dW_{k+1, \Pi^+} K_{k+1, \Pi^+} \mathcal{C}_{k+1, \Pi^+} \\ &\quad \chi_{k+1}(\Lambda_{k+1}) \exp \left(-S_{k+1}^+(\Lambda_{k+1}) + E_{k+1}(\Lambda_{k+1}) + R_{k+1, \Pi^+}^* + B_{k+1, \Pi^+}^* \right) \end{aligned} \quad (552)$$

3.18 final localization

The last expression is in final form except that the terms R_{k+1, Π^+}^* and B_{k+1, Π^+}^* are not properly localized and we need to establish some estimates. These are the problems to which we now turn. The proofs are very similar to the treatment in section 3.13.

Lemma 3.22. *The function R_{k, Π^+}^* can be written*

$$R_{k+1, \Pi^+}^* = \sum_{X \subset \Lambda_{k+1}} R_{k+1, \Pi^+}(X) + \sum_{X \in \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c), X \# \Lambda_{k+1}} B_{k+1, \Pi^+}^{*,(R)}(X) \quad (553)$$

The functions $R_{k+1, \Pi^+}(X, \Phi_{k+1})$ and $B_{k+1, \Pi^+}^{*,(R)}(X, \Phi_k, \Phi_{k+1})$ depend on the fields only in X , are analytic in $\mathcal{P}_{k+1}(\Lambda_{k+1}, 2\delta)$ and $\mathcal{P}_{k+1, \Omega^+}$ respectively, and on this domain they satisfy

$$\begin{aligned} |R_{k+1, \Pi^+}(X)| &\leq \lambda_{k+1}^{n_0} e^{-\kappa d_M(X)} \\ |B_{k+1, \Pi^+}^{*,(R)}(X)| &\leq \mathcal{O}(1) \lambda_{k+1}^{n_0} e^{-\kappa d_M(X, \text{mod } \Omega_{k+1}^c)} \end{aligned} \quad (554)$$

Proof. R_{k+1, Π^+}^* has many pieces, which we consider individually. Keep in mind that $\mathcal{P}_{k+1, \Omega^+} \subset \mathcal{P}_{k+1}(\Lambda_{k+1}, 2\delta)$.

The term $R_{k+1, \Pi^+}^{*(0)}$. This has the form $\sum_{X \subset \Lambda_{k+1}} R_{k+1, \Pi^+}^{*(0)}(X)$ with

$$R_{k+1, \Pi^+}^{*(0)}(X, \phi, \Phi_{k+1}) = (R_k^\#)_{L\Pi^+}(LX, \phi_L, \Phi_{k+1, L}) \quad \text{at } \phi = \phi_{k+1, \Omega'} \quad (555)$$

By lemma 3.20 we know $R_k^\#(X, \phi, \Phi_{k+1})$ is analytic in $\Phi_{k+1} \in \mathcal{P}_{k+1}^0(\Lambda_{k+1}, 2\delta)$ and $\phi \in \frac{1}{2}\mathcal{R}_k$. Hence the scaled version $R_{k+1, \mathbf{\Pi}^+}^{*(0)}(X, \phi, \Phi_{k+1})$ is analytic in $\Phi_{k+1} \in \mathcal{P}_{k+1}(\Lambda_{k+1}, 2\delta)$ and $\phi_L \in \frac{1}{2}\mathcal{R}_k$. Also by (133) $\Phi_{k+1} \in \mathcal{P}_{k+1, \mathbf{\Omega}^+}$ implies that $|\phi_{k+1, \mathbf{\Omega}'}| \leq C\lambda_{k+1}^{-\frac{1}{4}-2\delta}$ with similar bounds on the derivatives. Then $\phi_{k+1, \mathbf{\Omega}', L}$ and its derivatives satisfy the same bounds and since $2\delta < \epsilon$ these are more than sufficient to guarantee that $\phi_{k+1, \mathbf{\Omega}', L} \in \frac{1}{2}\mathcal{R}_k$. Thus $\mathcal{P}_{k+1, \mathbf{\Omega}^+}$ is a correct analyticity domain for $R_{k+1, \mathbf{\Pi}^+}^{*(0)}(X)$. From (511) and $d_{LM}(LX) = d_M(X)$ we have the bound on this domain

$$|R_{k+1, \mathbf{\Pi}^+}^{*(0)}(X)| \leq \mathcal{O}(1)L^3\lambda_k^{n_0}e^{-L(\kappa-6\kappa_0-6)d_M(X)} \quad (556)$$

Next localize by introducing the weakened field $\phi_{k+1, \mathbf{\Omega}'}(s)$ as before and define

$$R_{k+1, \mathbf{\Pi}^+}^{*(0)}(X, s) = R_{k+1, \mathbf{\Pi}^+}^{*(0)}(X, \phi_{k+1, \mathbf{\Omega}'}(s), \Phi_{k+1}) \quad (557)$$

which has the same bound. Now make a decoupling expansion roughly following the the treatment in lemma 3.15. This yields the strictly local expansion in $Z \in \mathcal{D}_{k+1}$

$$R_{k+1, \mathbf{\Pi}^+}^{*(0)} = \sum_{Z \cap \Lambda_{k+1} \neq \emptyset} (R_{k+1, \mathbf{\Pi}^+}^{*(0)})'(Z, \Phi_k, \Phi_{k+1}) \quad (558)$$

with the bound

$$|(R_{k+1, \mathbf{\Pi}^+}^{*(0)})'(Z)| \leq \mathcal{O}(1)L^3\lambda_k^{n_0}e^{-L(\kappa-8\kappa_0-8)d_M(Z)} \quad (559)$$

In the sum (558) consider terms with $Z \subset \Lambda_{k+1}$. These terms contribute to $R_{k+1, \mathbf{\Pi}^+}(Z)$. They only depend on Φ_{k+1} and so are analytic in $\mathcal{P}_{k+1}(\Lambda_{k+1}, 2\delta)$. We assume that κ is sufficiently large such that $L(\kappa - 8\kappa_0 - 8) \geq \kappa$. (It suffices for example that $\kappa \geq 16\kappa_0 + 16$). Then the exponent is dominated by $e^{-\kappa d_M(Z)}$. Furthermore for L sufficiently large and $n_0 \geq 4$

$$\mathcal{O}(1)L^3\lambda_k^{n_0} = \mathcal{O}(1)L^{3-n_0}\lambda_{k+1}^{n_0} \leq \frac{1}{6}\lambda_{k+1}^{n_0} \quad (560)$$

This is why we chose $n_0 \geq 4$. This is the basic mechanism which keeps the tiny terms tiny, in spite of the growth factor L^3 . Thus the bound is $|(R_{k+1, \mathbf{\Pi}^+}^{*(0)})'(Z)| \leq \frac{1}{6}\lambda_{k+1}^{n_0}e^{-\kappa d_M(Z)}$.

Now consider terms in (558) with $Z \# \Lambda_{k+1}$ which contribute to $B_{k+1, \mathbf{\Pi}^+}^{*,(R)}(X)$. We add connected components of Ω_{k+1}^c not disjoint with Z , and resum to polymers $X \in \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c)$. Each term is a contribution to $B_{k+1, \mathbf{\Pi}^+}^{*,(R)}(X)$ and is bounded by say $\mathcal{O}(1)L^3\lambda_k^{n_0}e^{-L(\kappa-9\kappa_0-9)d_M(X, \text{mod } \Omega_{k+1}^c)}$. See step (H.) in the proof of lemma 3.15 for details. Again for κ, L sufficiently large this is dominated by the required $\mathcal{O}(1)\lambda_{k+1}^{n_0}e^{-\kappa d_M(X, \text{mod } \Omega_{k+1}^c)}$.

The terms $R_{k+1, \mathbf{\Pi}^+}^{*(1)}$ We have $R_{k+1, \mathbf{\Pi}^+}^{*(1)} = \sum_{\square \subset \Lambda_{k+1}} R_{k+1, \mathbf{\Pi}^+}^{*(1)}(\square)$. where \square is an M -cube and

$$R_{k+1, \mathbf{\Pi}^+}^{*(1)}(\square) = S_{k+1}^*(\square, \Phi_{k+1}, \phi_{k+1, \mathbf{\Omega}'} - \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}) \quad (561)$$

Again (133) implies that $\phi_{k+1, \mathbf{\Omega}'}$ and $\phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}$ are bounded by $C\lambda_{k+1}^{-\frac{1}{4}-2\delta}$, also for derivatives. Therefore $|R_{k+1, \mathbf{\Pi}^+}^{*(1)}(\square)| \leq CM^3\lambda_{k+1}^{-\frac{1}{2}-4\delta}$.

Next introduce the abbreviated notation

$$\phi = \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)} \quad \delta\phi = \phi_{k+1, \mathbf{\Omega}'} - \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)} \quad (562)$$

Then $|\delta\phi| \leq C\lambda_{k+1}^{-\frac{1}{4}-2\delta} e^{-r_{k+1}}$ on Λ_{k+1} . This follows since $G_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}$ and $G_{k+1, \mathbf{\Omega}'}$ have random walk expansions which only differ outside Λ_{k+1}^* which is $[r_{k+1}]$ steps away from Λ_{k+1} . Then the representation

$$\begin{aligned} R_{k+1, \mathbf{\Pi}^+}^{*,(1)}(\square) &= S_{k+1}^*(\square, \Phi_{k+1}, \phi + \delta\phi) - S_{k+1}^*(\square, \Phi_{k+1}, \phi) \\ &= \frac{1}{2\pi i} \int_{|t|=e^{r_{k+1}}} \frac{dt}{t(t-1)} S_{k+1}(\square, \Phi_{k+1}, \phi + t\delta\phi) \end{aligned} \quad (563)$$

yields the bound

$$|R_{k+1, \mathbf{\Pi}^+}^{*,(1)}(\square)| \leq CM^3 \lambda_{k+1}^{-\frac{1}{2}-4\delta} e^{-r_{k+1}} \leq \lambda_{k+1}^{n_0+1} \quad (564)$$

For decoupling we have to be a little more careful, since there are two different Green's functions to decouple. First in $\delta\phi$ in $R_{k+1, \mathbf{\Pi}^+}^{*,(1)}(\square)$ we replace $\phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}, \phi_{k+1, \mathbf{\Omega}'}$ by truncated versions $\phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}^{\text{tr}}, \phi_{k+1, \mathbf{\Omega}'}^{\text{tr}}$ in which $G_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}, G_{k+1, \mathbf{\Omega}'}$ are replaced by

$$\begin{aligned} G_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}^{\text{tr}} &\equiv \sum_{\omega: X_{\omega_0} \subset \Lambda_{k+1}, X_{\omega} \cap \Lambda_{k+1}^{*,c} \neq \emptyset} G_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*), \omega} \\ G_{k+1, \mathbf{\Omega}'}^{\text{tr}} &\equiv \sum_{\omega: X_{\omega_0} \subset \Lambda_{k+1}, X_{\omega} \cap \Lambda_{k+1}^{*,c} \neq \emptyset} G_{k+1, \mathbf{\Omega}', \omega} \end{aligned} \quad (565)$$

Here the random walk is based on $\mathbf{\Omega}(\Lambda_{k+1}^*)$ in the first case, and on $\mathbf{\Omega}'$ in the second case. The condition $X_{\omega} \cap \Lambda_{k+1}^{*,c} \neq \emptyset$ is appropriate since terms with $X_{\omega} \subset \Lambda_{k+1}^*$ are the same for the two fields and so cancel. The fields $\phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}^{\text{tr}}, \phi_{k+1, \mathbf{\Omega}'}^{\text{tr}}$ now separately satisfy the $C\lambda_{k+1}^{-\frac{1}{4}-2\delta} e^{-r_{k+1}}$ bound.

Next we weaken the $\mathbf{\Omega}(\Lambda_{k+1}^*)$ fields by introducing parameters s for the $\mathbf{\Omega}(\Lambda_{k+1}^*)$ random walk and defining

$$\begin{aligned} R_{k+1, \mathbf{\Pi}^+}^{*,(1)}(\square, s) &= \frac{1}{2\pi i} \int_{|t|=e^{r_{k+1}}} \frac{dt}{t(t-1)} S_{k+1}(\square, \Phi_{k+1}, \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}(s) + t(\phi_{k+1, \mathbf{\Omega}'}^{\text{tr}} - \phi_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}^{\text{tr}}(s))) \end{aligned} \quad (566)$$

This also satisfies the bound (564). A decoupling expansion in \square^c leads to a sum of terms indexed by $Y \in \mathcal{D}_{k+1, \mathbf{\Omega}(\Lambda_{k+1}^*)}$. We resume to terms indexed by $Z \in \mathcal{D}_{k+1}$. As before this leads to the representation

$$R_{k+1, \mathbf{\Pi}^+}^{*,(1)} = \sum_{Z \cap \Lambda_{k+1} \neq \emptyset, Z \subset \Omega_{k+1}} (R_{k+1, \mathbf{\Pi}^+}^{*,(1)})'(Z, \Phi_{k+1}, \phi_{k+1, \mathbf{\Omega}'}^{\text{tr}}) \quad (567)$$

local in the indicated fields and (the coefficient 3κ achieved for M sufficiently large)

$$|(R_{k+1, \mathbf{\Pi}^+}^{*,(1)})'(Z)| \leq \mathcal{O}(1) \lambda_{k+1}^{n_0+1} e^{-3\kappa d_M(Z)} \quad (568)$$

Next we weaken the $\mathbf{\Omega}'$ field by introducing parameters s for the $\mathbf{\Omega}'$ random walk and defining for $X \in \mathcal{D}_{k+1}$

$$(R_{k+1, \mathbf{\Pi}^+}^{*,(1)})'(X, s) = (R_{k+1, \mathbf{\Pi}^+}^{*,(1)})'(X, \Phi_k, \phi_{k+1, \mathbf{\Omega}'}^{\text{tr}}(s)) \quad (569)$$

A decoupling expansion leads to a sum of terms indexed by $Y \in \mathcal{D}_{k+1, \mathbf{\Omega}'}$. We resume to terms indexed by $Z \in \mathcal{D}_{k+1}$. Then we have

$$R_{k+1, \mathbf{\Pi}^+}^{*,(1)} = \sum_{Z \cap \Lambda_{k+1} \neq \emptyset} (R_{k+1, \mathbf{\Pi}^+}^{*,(1)})''(Z, \Phi_k, \Phi_{k+1}) \quad (570)$$

local in the indicated fields and

$$|(R_{k+1, \mathbf{\Pi}^+}^{*(1)})''(Z)| \leq \mathcal{O}(1) \lambda_{k+1}^{n_0+1} e^{-2\kappa d_M(Z)} \quad (571)$$

Now split the terms into a contribution to $R_{k+1, \mathbf{\Pi}^+}(X)$ and $B_{k+1, \mathbf{\Pi}^+}^{*,(R)}(X)$ as in the previous case. For the terms $R_{k+1, \mathbf{\Pi}^+}(X)$ we use $\mathcal{O}(1) \lambda_{k+1}^{n_0+1} e^{-2\kappa d_M(X)} \leq \lambda_{k+1}^{n_0} e^{-\kappa d_M(X)}$

The terms $R_{k+1, \mathbf{\Pi}^+}^{*(2)}, R_{k+1, \mathbf{\Pi}^+}^{*(3)}$. Define $\phi, \delta\phi$ as before. We have the representations for $\square, X \subset \Lambda_{k+1}$

$$\begin{aligned} R_{k+1, \mathbf{\Pi}^+}^{*(2)}(\square) &= \frac{1}{2\pi i} \int_{|t|=e^{r_{k+1}}} \frac{dt}{t(t-1)} V_{k+1}(\square, \phi + t\delta\phi) \\ R_{k+1, \mathbf{\Pi}^+}^{*(3)}(X) &= \frac{1}{2\pi i} \int_{|t|=e^{r_{k+1}}} \frac{dt}{t(t-1)} E_{k+1}(X, \phi + t\delta\phi) \end{aligned} \quad (572)$$

which shows these are tiny. Then proceed with the localization and split as before.

The terms $R_{k+1, \mathbf{\Pi}^+}^{*(4)}, R_{k+1, \mathbf{\Pi}^+}^{*(5)}$. The term $R_{k+1, \mathbf{\Pi}^+}^{*(4)}$ is treated like $R_{k+1, \mathbf{\Pi}^+}^{*(1)}$, and $R_{k+1, \mathbf{\Pi}^+}^{*(5)}$ is treated like $R_{k+1, \mathbf{\Pi}^+}^{*(2)}$.

A difference is that these terms are initially localized in $\delta\Lambda_k = \Lambda_k - \Lambda_{k+1}$. In this domain $\delta\phi = \phi_{k+1, \mathbf{\Omega}'} - \phi_{k+1, \mathbf{\Omega}(\Lambda_k, \Omega_{k+1}, \Lambda_{k+1})}$ satisfies $|\delta\phi| \leq \mathcal{O}(1)e^{-r_{k+1}}$. This is because the random walk expansions for $G_{k+1, \mathbf{\Omega}'}$ and $G_{k+1, \mathbf{\Omega}(\Lambda_k, \Omega_{k+1}, \Lambda_{k+1})}$ starting in $\delta\Lambda_k$ only differ outside $\Lambda_{k+1}^{c,*}$, i.e. in Λ_{k+1}^{\natural} . Hence they have at least $[r_{k+1}]$ steps and this gives the tiny factor $e^{-r_{k+1}}$.

Another point is that in the expression for say $(R_{k+1, \mathbf{\Pi}^+}^{*(5)})''(Z)$, only paths which stay in Z contribute. If $Z \cap \Lambda_{k+1} = \emptyset$ then the paths must stay in Λ_{k+1}^c as well as visiting Λ_{k+1}^{\natural} . Hence there are no paths contributing to $\delta\phi$ in this case, from which one can deduce $(R_{k+1, \mathbf{\Pi}^+}^{*(5)})'(Z) = 0$. Thus we can restrict to $Z \cap \Lambda_{k+1} \neq \emptyset$ and hence $Z \# \Lambda_{k+1}$. All these terms contribute to $B_{k+1, \mathbf{\Pi}^+}^{*,(R)}(X)$.

Lemma 3.23. *The function $B_{k+1, \mathbf{\Pi}^+}^*$ can be written in the form*

$$B_{k+1, \mathbf{\Pi}^+}^* = \sum_{X \in \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c), X \# \Lambda_{k+1}} (B_{k+1, \mathbf{\Pi}^+}^*)'(X) \quad (573)$$

where $(B_{k+1, \mathbf{\Pi}^+}^*)'(X, \Phi_{k+1, \mathbf{\Omega}^+}, W_{k+1, \mathbf{\Pi}^+})$ depends on the fields only in X , is analytic in

$$\Phi_{k+1, \mathbf{\Omega}^+} \in \mathcal{P}_{k+1, \mathbf{\Omega}^+} \quad |W_j| \leq B_w p_j L^{\frac{1}{2}(k+1-j)} \quad j = 0, 1, \dots, k \quad (574)$$

On this domain it satisfies

$$|(B_{k+1, \mathbf{\Pi}^+}^*)'(X)| \leq \frac{1}{2} B_0 \lambda_{k+1}^\beta e^{-\kappa d_M(X, \text{mod } \Omega_{k+1}^c)} \quad (575)$$

Proof. $B_{k+1, \mathbf{\Pi}^+}^*$ has two parts which we consider separately.

The term $B_{k+1, \mathbf{\Pi}^+}^{(0)}$ $[(B_k^\#)_{L^{-1}}]_{\mathbf{\Pi}^+}(\Lambda_{k+1})$. This has the form $B_{k+1, \mathbf{\Pi}^+}^{(0)} = \sum_X B_{k+1, \mathbf{\Pi}^+}^{(0)}(X)$ with the sum over $X \in \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c), X \# \Lambda_{k+1}$ and

$$B_{k+1, \mathbf{\Pi}^+}^{(0)}(X, \phi, \Phi_{k+1, \mathbf{\Omega}^+}, W_{k+1, \mathbf{\Pi}^+}) = [B_k^\#]_{L\mathbf{\Pi}^+}(LX, \phi_L, \Phi_{k+1, \mathbf{\Omega}^+, L}, W_{k+1, \mathbf{\Pi}^+, L}) \quad (576)$$

evaluated at $\phi = \phi_{k+1, \mathbf{\Omega}'}$. By lemma 3.17 $B_k^\#$ is analytic in the domain $\phi \in \frac{1}{2}\mathcal{R}_k$ and (498). Hence $B_{k+1, \mathbf{\Pi}^+}^{(0)}$ is analytic in $\phi_L \in \frac{1}{2}\mathcal{R}_k$ and (574) which is the scaling of (498). Furthermore (574) and (133) imply $\phi_{k, \mathbf{\Omega}', L} \in \frac{1}{2}\mathcal{R}_k$, hence with $\phi = \phi_{k+1, \mathbf{\Omega}'}$ we are in the analyticity domain for the function. Also from lemma 3.17 since $\beta < \frac{1}{4} - 10\epsilon$ and $\lambda_k < \lambda_{k+1}$ we have the bound $|B_{k, \mathbf{\Pi}^+}^\#(X)| \leq \mathcal{O}(1)L^3\lambda_{k+1}^\beta e^{-L(\kappa-6\kappa_0-6)d_{LM}(X, \text{mod } \Omega_{k+1}^c)}$. Since $d_{LM}(LX, \text{mod } L\Omega_{k+1}^c) = d_M(X, \text{mod } \Omega_{k+1}^c)$ we have on (574)

$$|B_{k+1, \mathbf{\Pi}^+}^{(0)}(X)| \leq \mathcal{O}(1)L^3\lambda_{k+1}^\beta e^{-L(\kappa-6\kappa_0-6)d_M(X, \text{mod } \Omega_{k+1}^c)} \quad (577)$$

To localize we weaken the coupling by again introducing the field $\phi_{k+1, \mathbf{\Omega}'}(s)$ where $s = \{s_\square\}$ is indexed by cubes \square compatible with $\mathbf{\Omega}' = (\mathbf{\Omega}(\Lambda_k^*), \Omega_{k+1})$ and not in X . This includes cubes in a connected component of Ω_{k+1}^c disjoint from X , but not cubes in a connected component of Ω_{k+1}^c contained in X . Now define

$$B_{k+1, \mathbf{\Pi}^+}^{*(0)}(X, s) = B_{k+1, \mathbf{\Pi}^+}^{*(0)}(X, \phi_{k+1, \mathbf{\Omega}'}(s), \Phi_{k+1, \mathbf{\Omega}^+}, W_{k+1, \mathbf{\Pi}^+}) \quad (578)$$

which has the same analyticity domain and the same bound. Now make a decoupling expansion similar to lemma 3.17. This generates terms indexed by multiscale polymers $Y \in \mathcal{D}_{k+1, \mathbf{\Omega}'}$. These are resummed to give terms indexed by polymers $Z_0 \in \mathcal{D}_{k+1}$. We take the union with any new connected components of Ω_{k+1}^c to get $Z = Z_0^+ \in \mathcal{D}_{k+1}(\text{mod } \Omega_{k+1}^c)$. We then find that

$$B_{k+1, \mathbf{\Pi}^+}^{*(0)} = \sum_{Z \# \Lambda_{k+1}} (B_{k+1, \mathbf{\Pi}^+}^{*(0)})'(Z, \Phi_{k+1, \mathbf{\Omega}^+}, W_{k+1, \mathbf{\Pi}^+}) \quad (579)$$

In estimating this we use the bound

$$|Z_0 - X|_M + d_M(X, \text{mod } \Omega_{k+1}^c) \geq d_M(Z, \text{mod } \Omega_{k+1}^c) \quad (580)$$

which is proved as in (483). Using also (611) we find that

$$|(B_{k+1, \mathbf{\Pi}^+}^{*(0)})'(Z)| \leq \mathcal{O}(1)L^3\lambda_{k+1}^\beta e^{-L(\kappa-8\kappa_0-8)d_M(Z, \text{mod } \Omega_{k+1}^c)} \leq \frac{1}{4}B_0\lambda_{k+1}^\beta e^{-\kappa d_M(Z, \text{mod } \Omega_{k+1}^c)} \quad (581)$$

We have assumed B_0 is sufficiently large so that $\mathcal{O}(1)L^3 \leq \frac{1}{4}B_0$

The term $B_{k+1, \mathbf{\Pi}^+}^{*,(1)}$. This term depends on

$$E_k^*(X, \phi) \equiv (\mathcal{B}E_k + E_k^\#)_{L^{-1}}(X, \phi) = (\mathcal{B}E_k + E_k^\#)(LX, \phi_L) \quad (582)$$

For $\phi \in \frac{1}{2}\mathcal{R}_k$ we have $|E_k(X, \phi)| \leq \lambda_k^\beta e^{-\kappa d_M(X)}$ and $|E_k^\#(X, \phi)| \leq \mathcal{O}(1)L^3\lambda_k^{\frac{1}{4}-10\epsilon} e^{-L(\kappa-3\kappa_0-3)d_{LM}(X)}$. Hence for $\phi_L \in \frac{1}{2}\mathcal{R}_k$

$$|E_k^*(X, \phi)| \leq \mathcal{O}(1)L^3\lambda_{k+1}^\beta e^{-2\kappa d_M(X)} \quad (583)$$

and the same holds in the smaller domain $\phi \in \mathcal{R}_{k+1}$.

Now $B_{k+1, \mathbf{\Pi}^+}^{*,(1)} = \sum_X B_{k+1, \mathbf{\Pi}^+}^{*,(1)}(X)$ where the sum is over $X \in \mathcal{S}$ and $X \# \Lambda_{k+1}$ and

$$\begin{aligned} B_{k+1, \mathbf{\Pi}^+}^{*,(1)}(X, \phi_{k+1, \mathbf{\Omega}'}) &= (\mathcal{T}_{\Lambda_{k+1}} E_k^*)(X, \phi_{k+1, \mathbf{\Omega}'}) \\ &= \alpha_0(E_k^*, X) \text{Vol}(\Lambda_{k+1} \cap X) + \alpha_2(E_k^*, X) \|\phi_{k+1, \mathbf{\Omega}'}\|_{X \cap \Lambda_{k+1}}^2 \\ &\quad + \sum_{\mu} \alpha_{2, \mu}(E_k^*, X) \int_{X \cap \Lambda_{k+1}} \phi_{k+1, \mathbf{\Omega}'} \partial_{\mu} \phi_{k+1, \mathbf{\Omega}'} \end{aligned} \quad (584)$$

The bound (583) implies (see similar estimates in part I)

$$\begin{aligned} |\alpha_0(E_k^*, X)| &\leq \mathcal{O}(1) \text{Vol}(X)^{-1} L^3 \lambda_{k+1}^\beta e^{-2\kappa d_M(X)} \\ |\alpha_2(E_k^*, X)| &\leq \mathcal{O}(1) \text{Vol}(X)^{-1} L^3 \lambda_{k+1}^{\beta+\frac{1}{2}+6\epsilon} e^{-2\kappa d_M(X)} \\ |\alpha_{2,\mu}(E_k^*, X)| &\leq \mathcal{O}(1) \text{Vol}(X)^{-1} L^3 \lambda_{k+1}^{\beta+\frac{1}{2}+5\epsilon} e^{-2\kappa d_M(X)} \end{aligned} \quad (585)$$

Furthermore $|\phi_{k+1, \mathbf{\Omega}'}| \leq \mathcal{O}(1) \lambda_{k+1}^{-\frac{1}{4}-2\delta}$ and hence $\|\phi_{k+1, \mathbf{\Omega}'}\|_{X \cap \Lambda_{k+1}}^2 \leq \mathcal{O}(1) \text{Vol}(X \cap \Lambda_{k+1}) \lambda_{k+1}^{-\frac{1}{2}-4\delta}$. The same bound holds for $\int_{X \cap \Lambda_{k+1}} \phi_{k+1, \mathbf{\Omega}'} \partial_\mu \phi_{k+1, \mathbf{\Omega}'}$. Therefore since $2\delta < \epsilon$

$$|B_{k+1, \mathbf{\Pi}^+}^{*(1)}(X)| \leq \mathcal{O}(1) L^3 \lambda_{k+1}^\beta e^{-2\kappa d_M(X)} \quad (586)$$

Note that $X \in \mathcal{S}$ and $X \# \Lambda_{k+1}$ rules out that X contains any connected components of Ω_{k+1}^c so we can replace the $d_M(X)$ by $d_M(X, \text{mod } \Omega_{k+1}^c)$.

Now localize as before and get a sum over strictly localized functions $(B_{k+1, \mathbf{\Pi}^+}^{*(1)})'(Z, \Phi_k, \Phi_{k+1})$ satisfying

$$|(B_{k+1, \mathbf{\Pi}^+}^{*(1)})'(Z)| \leq \frac{1}{4} B_0 \lambda_{k+1}^\beta e^{-\kappa d_M(Z, \text{mod } \Omega_{k+1}^c)} \quad (587)$$

The lemma now holds with $(B_{k+1, \mathbf{\Pi}^+}^*)'(Z) = (B_{k+1, \mathbf{\Pi}^+}^{*(0)})'(Z) + (B_{k+1, \mathbf{\Pi}^+}^{*(1)})'(Z)$.

Conclusion: From the last two lemmas we can write

$$R_{k+1, \mathbf{\Pi}^+}^* + B_{k+1, \mathbf{\Pi}^+}^* = R_{k+1, \mathbf{\Pi}^+}(\Lambda_{k+1}) + B_{k+1, \mathbf{\Pi}^+}(\Lambda_{k+1}) \quad (588)$$

where

$$B_{k+1, \mathbf{\Pi}^+}(X) = B_{k+1, \mathbf{\Pi}^+}^{*,(R)}(X) + (B_{k+1, \mathbf{\Pi}^+}^*)'(X) \quad (589)$$

Then $B_{k+1, \mathbf{\Pi}^+}(X)$ is analytic in the domain (574) and by (554) and (575) satisfies there

$$|B_{k+1, \mathbf{\Pi}^+}(X)| \leq \left(\mathcal{O}(1) \lambda_{k+1}^{n_0} + \frac{1}{2} B_0 \lambda_{k+1}^\beta \right) e^{-\kappa d_M(X, \text{mod } \Omega_{k+1}^c)} \leq B_0 \lambda_{k+1}^\beta e^{-\kappa d_M(X, \text{mod } \Omega_{k+1}^c)} \quad (590)$$

Also note that $\tilde{B}_{k+1, \mathbf{\Pi}^+}(\Lambda_k, \Lambda_{k+1})$ is analytic on (574) and satisfies for B_0 sufficiently large

$$|\tilde{B}_{k+1, \mathbf{\Pi}^+}(\Lambda_k, \Lambda_{k+1})| \leq B_0 |\Lambda_k^{(k+1)} - \Lambda_{k+1}^{(k+1)}| \quad (591)$$

Finally substituting (588) into (552) yields

$$\begin{aligned} \rho_{k+1}(\Phi_{k+1}) &= Z_{k+1} \sum_{\mathbf{\Pi}^+} \int d\Phi_{k+1, \mathbf{\Omega}^{+,c}} dW_{k+1, \mathbf{\Pi}^+} K_{k+1, \mathbf{\Pi}^+} \mathcal{C}_{k+1, \mathbf{\Pi}^+} \\ &\chi_{k+1}(\Lambda_{k+1}) \exp \left(-S_{k+1}^+(\Lambda_{k+1}) + E_{k+1}(\Lambda_{k+1}) + R_{k+1, \mathbf{\Pi}^+}(\Lambda_{k+1}) + B_{k+1, \mathbf{\Pi}^+}(\Lambda_{k+1}) \right) \end{aligned} \quad (592)$$

All properties of the various functions have been established, so this completes the induction and the proof of the main theorem.

In the third and final paper (which is much shorter) we establish the convergence of the expansion and prove the stability bound.

A notation

For various reasons we have deviated from the notation employed by Balaban in [8]- [14]. The following table is a dictionary for connecting those papers with the present paper. It is not exact.

This work	Balaban
Ω_k	Ω_k
Λ_k	Ω_k'' or Z_k^c
Λ_k^*	W_k
$\delta\Omega_k^{(k)}$	Λ_k
Π	\mathbb{A}
$\Omega(\Lambda_k^*)$	$\mathbb{B}(W_k)$
$\Omega' = (\Omega(\Lambda_k^*), \Omega_{k+1})$	$(\mathbb{B}_k(W_k) \cap \Omega_{k+1}^c, \Omega_{k+1})$
$\Phi_{k,\Omega}$	ψ
$W_{k,\Omega}$	ψ'
$\Phi_{k,\Omega_{k+1}}$	θ
$\hat{\Phi}_{k+1,\Omega'}$	$\tilde{\theta}$
$\hat{\Psi}_{k,\Omega'}$	$\psi^{(k)}(\tilde{\theta})$

Table 1: comparison of notation

B a lattice identity

Λ be a union of cubes in \mathbb{T}_{M+N-k}^{-k} as in the text, and let f, g be functions on a neighborhood of Λ . We prove the following identity:

Theorem B.1.

$$\langle \partial f, \partial g \rangle_{*,\Lambda} = \langle (-\Delta)f, g \rangle_{\Lambda} + \frac{1}{2} \sum_{x \in \Lambda, x' \in \Lambda^c} L^{-2k} \partial f(x, x') (g(x) + g(x')) \quad (593)$$

Proof. We have

$$\begin{aligned} \langle \partial f, \partial g \rangle_{*,\Lambda} &= \sum_{\langle x, x' \rangle \in \Lambda} L^{-3k} \partial f(x, x') \partial g(x, x') + \frac{1}{2} \sum_{x \in \Lambda, x' \in \Lambda^c} L^{-3k} \partial f(x, x') \partial g(x, x') \\ &= \sum_{\langle x, x' \rangle \in \Lambda} L^{-3k} \partial f(x, x') \partial g(x, x') + \frac{1}{2} \sum_{x \in \Lambda, x' \in \Lambda^c} L^{-2k} \partial f(x, x') (g(x') - g(x)) \end{aligned} \quad (594)$$

The first sums are over oriented bonds $\langle x, x' \rangle = \langle x, x + L^{-k} e_\mu \rangle$. The first line is the definition and the second line follows by $\partial g(x, x') = L^k(g(x') - g(x))$.

On the other hand if g_Λ is the restriction to Λ we have

$$\begin{aligned} \langle (-\Delta)f, g \rangle_\Lambda &= \langle (-\Delta)f, g_\Lambda \rangle = \langle \partial f, \partial g_\Lambda \rangle = \sum_{\langle x, x' \rangle \in \Lambda} L^{-3k} \partial f(x, x') \partial g(x, x') \\ &+ \sum_{\langle x, x' \rangle : x \in \Lambda, x' \in \Lambda^c} L^{-3k} \partial f(x, x') \partial g_\Lambda(x, x') + \sum_{\langle x, x' \rangle : x' \in \Lambda, x \in \Lambda^c} L^{-3k} \partial f(x, x') \partial g_\Lambda(x, x') \end{aligned} \quad (595)$$

In the last sum $\partial g_\Lambda(x, x') = L^k g(x')$ and in the previous sum $\partial g_\Lambda(x, x') = -L^k g(x)$. Therefore the last two sums are

$$\begin{aligned} &- \sum_{\langle x, x' \rangle : x \in \Lambda, x' \in \Lambda^c} L^{-2k} \partial f(x, x') g(x) + \sum_{\langle x, x' \rangle : x' \in \Lambda, x \in \Lambda^c} L^{-2k} \partial f(x, x') g(x') \\ &= - \sum_{x \in \Lambda, x' \in \Lambda^c} L^{-2k} \partial f(x, x') g(x) \end{aligned} \quad (596)$$

Here in the second sum over we have relabeled $x \leftrightarrow x'$ and used $\partial f(x', x) = -\partial f(x, x')$. Then this sum and the one preceding it are sums of the same function over outward bonds $x' \in \Lambda, x \in \Lambda^c$. But the first sum comes from outward bonds so $\langle x, x' \rangle$ is oriented and the second sum comes from outward bonds so $\langle x', x \rangle$ is oriented. We combine them into an unrestricted sum over all outward bonds to get the last line.

Now $\langle \partial f, \partial g \rangle_{*, \Lambda} - \langle (-\Delta)f, g \rangle_\Lambda$ gives the surface integral in (593) as announced.

C another identity

We seek an alternate expression for

$$C_{k, \mathbf{\Omega}^+, r} = \left[\Delta_{k, \mathbf{\Omega}} + \frac{a}{L^2} Q^T Q + r \right]_{\Omega_{k+1}}^{-1} \quad (597)$$

where as in the text $\mathbf{\Omega}^+ = (\mathbf{\Omega}, \Omega_{k+1}) = (\Omega_1, \dots, \Omega_k, \Omega_{k+1})$.

Lemma C.1.

$$C_{k, \mathbf{\Omega}^+, r} = \left[A_{k, r} + a_k^2 A_{k, r} Q_k G_{k, \mathbf{\Omega}^+, r} Q_k^T A_{k, r} \right]_{\Omega_{k+1}} \quad (598)$$

where

$$\begin{aligned} A_{k, r} &= \frac{1}{a_k + r} (I - Q^T Q) + \frac{1}{a_k + aL^{-2} + r} Q^T Q \\ B_{k, r} &= \frac{r}{a_k + r} (I - Q^T Q) + \frac{aL^{-2} + r}{a_k + aL^{-2} + r} Q^T Q \\ G_{k, \mathbf{\Omega}^+, r} &= \left[-\Delta + \bar{\mu}_k + \left[Q_{k, \mathbf{\Omega}}^T \mathbf{a} Q_{k, \mathbf{\Omega}} \right]_{\Omega_{k+1}^c} + a_k \left[Q_k^T B_{k, r} Q_k \right]_{\Omega_{k+1}} \right]_{\Omega_1}^{-1} \end{aligned} \quad (599)$$

Proof. Start with

$$\exp \left(\frac{1}{2} \langle f, C_{k, \mathbf{\Omega}^+, r} f \rangle \right) = \text{const} \int d\Phi \exp \left(\langle \Phi, f \rangle - \frac{a}{2L^2} \|Q\Phi\|^2 - \frac{r}{2} \|\Phi\|^2 - \frac{1}{2} \langle \Phi, \Delta_{k, \mathbf{\Omega}} \Phi \rangle \right) \quad (600)$$

where $f, \Phi : \Omega_{k+1}^{(k)} \rightarrow \mathbb{R}$. In general for $\phi : \Omega_1 \rightarrow \mathbb{R}$

$$\begin{aligned} & \exp\left(-\frac{1}{2} \langle \Phi_{k,\mathbf{\Omega}}, \Delta_{k,\mathbf{\Omega}} \Phi_{k,\mathbf{\Omega}} \rangle\right) \\ &= \text{const} \int \exp\left(-\frac{1}{2} \|\mathbf{a}^{1/2}(\Phi_{k,\mathbf{\Omega}} - Q_{k,\mathbf{\Omega}}\phi)\|^2 - \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k)\phi \rangle\right) d\phi \end{aligned} \quad (601)$$

This follows from (44), (47), (49) with $\phi_{\Omega_1^c} = 0$. Specializing to $\Phi_{k,\mathbf{\Omega}} = (0, \Phi)$ with Φ on Ω_{k+1} this says

$$\begin{aligned} & \exp\left(-\frac{1}{2} \langle \Phi, \Delta_{k,\mathbf{\Omega}} \Phi \rangle\right) \\ &= \text{const} \int \exp\left(-\frac{a_k}{2} \|\Phi - Q_k\phi\|_{\Omega_{k+1}}^2 - \frac{1}{2} \|\mathbf{a}^{1/2} Q_{k,\mathbf{\Omega}} \phi\|_{\Omega_{k+1}^c}^2 - \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k)\phi \rangle\right) d\phi \end{aligned} \quad (602)$$

Insert (602) into (600) and do the integral over Φ which is

$$\begin{aligned} & \int d\Phi \exp\left(\langle \Phi, f \rangle - \frac{a}{2L^2} \|Q\Phi\|^2 - \frac{r}{2} \|\Phi\|^2 - \frac{a_k}{2} \|\Phi - Q_k\phi\|^2\right) \\ &= \int d\Phi \exp\left(\langle \Phi, f + a_k Q_k \phi \rangle - \frac{1}{2} \langle \Phi, (a_k + r + aL^{-2} Q_k^T Q_k) \Phi \rangle - \frac{a_k}{2} \|Q_k \phi\|^2\right) \\ &= \text{const} \exp\left(\frac{1}{2} \langle (f + a_k Q_k), A_{k,r}(f + a_k Q_k) \rangle - \frac{a_k}{2} \|Q_k \phi\|^2\right) \end{aligned} \quad (603)$$

Here we used $(a_k + r + aL^{-2} Q_k^T Q_k)^{-1} = A_{k,r}$ which follows since $Q_k^T Q_k$ is a projection. Now we have

$$\begin{aligned} & \exp\left(\frac{1}{2} \langle f, C_{k,\mathbf{\Omega}^+,r} f \rangle\right) \\ &= \text{const} \int \exp\left(\frac{1}{2} \langle (f + a_k Q_k \phi), A_{k,r}(f + a_k Q_k \phi) \rangle\right. \\ & \quad \left. - \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k + a_k Q_k^T Q_k + [Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}}]_{\Omega_{k+1}^c}) \phi \rangle\right) d\phi \\ &= \text{const} \exp\left(\frac{1}{2} \langle f, A_{k,r} f \rangle\right) \int \exp\left(\langle \phi, a_k Q_k^T A_{k,r} f \rangle\right. \\ & \quad \left. - \frac{1}{2} \langle \phi, (-\Delta + \bar{\mu}_k + a_k Q_k^T B_{k,r} Q_k + [Q_{k,\mathbf{\Omega}}^T \mathbf{a} Q_{k,\mathbf{\Omega}}]_{\Omega_{k+1}^c}) \phi \rangle\right) d\phi \\ &= \text{const} \exp\left(\frac{1}{2} \langle f, A_{k,r} f \rangle\right) \int \exp\left(\langle \phi, a_k Q_k^T A_{k,r} f \rangle - \frac{1}{2} \langle \phi, G_{k,\mathbf{\Omega}^+,r}^{-1} \phi \rangle\right) d\phi \\ &= \text{const} \exp\left(\frac{1}{2} \langle f, A_{k,r} f \rangle + \frac{a_k^2}{2} \langle f, A_{k,r} Q_k G_{k,\mathbf{\Omega}^+,r} Q_k^T A_{k,r} f \rangle\right) \end{aligned} \quad (604)$$

which gives the result. Here we have used the identity

$$\begin{aligned} a_k^2 A_{k,r} - a_k &= a_k^2 \left(\frac{1}{a_k + r} (I - Q^T Q) + \frac{1}{a_k + aL^{-2} + r} Q^T Q \right) - a_k \\ &= a_k \left(\left(\frac{a_k}{a_k + r} - 1 \right) (I - Q^T Q) + \left(\frac{a_k}{a_k + aL^{-2} + r} - 1 \right) Q^T Q \right) \\ &= -a_k \left(\frac{r}{a_k + r} (I - Q^T Q) + \frac{aL^{-2} + r}{a_k + aL^{-2} + r} Q^T Q \right) \\ &= -a_k B_{k,r} \end{aligned} \quad (605)$$

D connected polymer sums

Let $X \in \mathcal{D}_{k,\Omega}$ be a multiscale polymer, and let $|X|_\Omega$ be the number of elementary cubes in X as in section 3.1.2, except now we work in arbitrary dimension d .

Lemma D.1. *For κ_* sufficiently large ($\kappa_* \geq \mathcal{O}(\log L)$) and any elementary cube in $\square \subset \mathcal{D}_{k,\Omega}$*

$$\sum_{X \in \mathcal{D}_{k,\Omega}, X \supset \square} e^{-\kappa_* |X|_\Omega} \leq e^{-\frac{1}{2}\kappa_*} \quad (606)$$

Proof. We have

$$\sum_{X \supset \square} e^{-\kappa_* |X|_\Omega} \leq \sum_{n \geq 1} e^{-\kappa_* n} |\{X \supset \square : |X|_\Omega = n\}| \quad (607)$$

To count $|\{X \supset \square : |X|_\Omega = n\}|$ note that for each such X there is a tree (not unique) whose lines are pairs of adjacent cubes in X . This tree will have $n - 1$ lines. Distinct polymers give distinct trees so that number is less than the number of such trees. Each tree can be traversed with a path starting at \square and traversing each line exactly twice. Thus the number of trees is less than the number of paths of length $2n$ starting at \square . Since each cube has at most $2dL^{d-1}$ neighbors this is bounded by $(2dL^{d-1})^{2n}$. Thus the sum is bounded by

$$\sum_{n \geq 1} e^{-\kappa_* n} (2dL^{d-1})^{2n} \leq \sum_{n \geq 1} e^{-(\kappa_* - 2 \log(2dL^{d-1}))n} \leq e^{-\frac{1}{2}\kappa_*} \quad (608)$$

provided $\frac{1}{2}\kappa_* \geq 2 \log(2dL^{d-1}) + \log 2$. This completes the proof.

For the next result we relax the condition that X be connected, so X is just a union of elementary cubes: $L^{-(k-j)}M$ cubes in $\delta\Omega_j$ (M -cubes in Ω_k). We sum over $Y \supset X$ of the same form. A connected component of Y has the property that every connected component of X is either contained in it or is disjoint from it.

Lemma D.2. *With κ_* as above*

$$\sum_{Y \supset X}' e^{-\kappa_* |Y-X|_\Omega} \leq \exp\left(e^{-\frac{1}{2}\kappa_*} (2dL^{d-1} + 1) |X|_\Omega\right) \quad (609)$$

where the primed sum means every connected component of Y contains at least one connected component of X

Proof. Let $\{W_\alpha\}$ be the connected components of $Y - X$. Let X' be the enlargement of X formed by adding all cubes which have a face in common with X . Then each W_α contains some cube in $X' - X$, (If not W_α is disjoint from X . Then W_α is a connected subset of Y with no path in Y to any other cube in Y , since any such path would have to pass through X . Hence W_α is a connected component of Y which contains no connected component of X which is a contradiction.) Conversely let $\{W_\alpha\}$ be a collection of disjoint connected subsets in X^c with the property that each contains a cube in $X' - X$. Then $Y = X \cup (\cup_\alpha W_\alpha)$ has the property that every connected component of Y contains at least one connected component of X . (It contains either a connected component of X or some W_α . In the latter case the cube in W_α provides a link to some connected component of X which is therefore included.) The upshot is that the sum can be written as a sum over the $\{W_\alpha\}$.

We enlarge the sum to a sum over collections $\{\square_\alpha\}$ of disjoint cubes in $X' - X$ or even X' , and connected W_α so $W_\alpha \supset \square_\alpha$. Using also the previous lemma the sum is dominated by

$$\begin{aligned} & \sum_{\{\square_\alpha\}} \sum_{\{W_\alpha: W_\alpha \supset \square_\alpha\}} e^{-\kappa_* \sum_\alpha |W_\alpha|} \Omega = \sum_{\{\square_\alpha\}} \prod_\alpha \sum_{W \supset \square_\alpha} e^{-\kappa_* |W|} \Omega \\ & \leq \sum_{\{\square_\alpha\}} \prod_\alpha e^{-\frac{1}{2}\kappa_*} = \left(1 + e^{-\frac{1}{2}\kappa_*}\right)^{|X'|} \Omega \leq \exp\left(e^{-\frac{1}{2}\kappa_*} |X'| \Omega\right) \end{aligned} \quad (610)$$

Since $|X'| \Omega \leq (2dL^{d-1} + 1)|X| \Omega$ we have the result.

Remark. A variation is the following. Suppose $X \in \mathcal{D}_k(\text{mod } \Omega_k)$, so X is connected and either contains a connected component of Ω_k^c or is disjoint from it. Then $X' - X \subset \Omega_k$ so the anchors $\{\square_\alpha\}$ in $X' - X$ are M -cubes. Hence we get $|X - X'|_M \leq |X'|_M$ rather than the (possibly much larger) $|X'| \Omega$. The result is

$$\sum_{Y \in \mathcal{D}_{k, \Omega}} e^{-\kappa_* |Y - X|} \Omega \leq \exp\left(e^{-\frac{1}{2}\kappa_*} (2dL^{d-1} + 1) |X|_M\right) \quad (611)$$

E disconnected polymer sums

Let Y be a collection of M -blocks \square in a lattice of dimension d . Y is not necessarily connected. We define various lengths $\ell(Y)$, $\ell'(Y)$, $\tilde{\ell}(Y)$ associated with Y . In the following "tree" means continuum tree.

1. $M\ell'_M(Y)$ is the length of a minimal tree whose vertices are the centers of the blocks in Y .
2. $M\ell_M(Y)$ is the length of a minimal tree whose vertices are one point from each block in Y .
3. $M\tilde{\ell}_M(Y)$ is the length of a minimal tree whose vertices are one point from each block in Y and possibly other points.

We have trivially $\tilde{\ell}_M(Y) \leq \ell_M(Y) \leq \ell'_M(Y)$. If Y is connected then $\tilde{\ell}_M(Y)$ differs slightly from $d_M(Y)$ defined in part I, since the latter requires a minimal tree to lie in Y . But we do have $\tilde{\ell}_M(Y) \leq d_M(Y)$. Recall also that $|Y|_M$ is the number of M -blocks in Y .

Lemma E.1.

1. $\ell_M(Y) \leq 2\tilde{\ell}_M(Y)$
2. $\ell'_M(Y) \leq \ell_M(Y) + \sqrt{d}|Y|_M$
3. $|Y|_M \leq 4(2^d + 1)(\ell_M(Y) + 1)$

Proof. We can take $M = 1$ and drop the subscript M .

1. [20]. Let $\tilde{\tau}$ be a minimal tree of length $\ell(\tilde{\tau}) = \tilde{\ell}(Y)$. One can traverse $\tilde{\tau}$ with a path $\tilde{\gamma}$ which passes through every vertex and has length $2\tilde{\ell}_M(Y)$. The path runs through the vertices in some order v_1, \dots, v_n . Replace the segment from the vertex v_i to the vertex v_{i+1} by a straight line. This gives a path γ which passes thru each vertex exactly once and is shorter than $\tilde{\gamma}$. Hence $\ell(Y) \leq \ell(\gamma) \leq \ell(\tilde{\gamma}) = 2\tilde{\ell}(Y)$

2. Let τ be a minimal tree on the blocks of Y of length $\ell(\tau) = \ell(Y)$. Replace each line by a line from center to center and call the resulting tree τ' . This increases the length of each line by at most $2\sqrt{(\frac{1}{2})^2 + \dots + (\frac{1}{2})^2} = \sqrt{d}$. Then we have

$$\ell'(Y) \leq \ell(\tau') \leq \ell(\tau) + \sqrt{d}|Y| = \ell(Y) + \sqrt{d}|Y| \quad (612)$$

3. Given Y construct a path γ which goes through each vertex exactly once and has length $\ell(\gamma) \leq 2\ell(Y)$ as in part (1.). Let γ_1 be the first $2^d + 1$ lines, let γ_2 be the next $2^d + 1$ lines, etc., and let γ_n be the last $2^d + 1$ or fewer lines, so $\gamma = \gamma_1 \cup \dots \cup \gamma_n$. For $1 \leq i \leq (n-1)$ we must have $\ell(\gamma_i) \geq 1$ since at most 2^d blocks be mutually touching. Then if $|\gamma|$ is the number of lines in the path γ

$$\begin{aligned} |Y| - 1 = |\gamma| &= \sum_{i=1}^n |\gamma_i| \leq n(2^d + 1) \leq \left(\sum_{i=1}^{n-1} \ell(\gamma_i) + 1 \right) 2(2^d + 1) \\ &\leq (\ell(\gamma) + 1) 2(2^d + 1) \leq 4(2^d + 1)\ell(Y) + 2(2^d + 1) \end{aligned} \quad (613)$$

which is sufficient.

Lemma E.2.

1. There are constants $a', b' = \mathcal{O}(1)$ such that for any M -cube \square_0

$$\sum_{Y: Y \supset \square_0} \exp(-a' \ell'_M(Y)) \leq b' \quad (614)$$

2. There are constants $a, b = \mathcal{O}(1)$ such that

$$\sum_{Y: Y \supset \square_0} \exp(-a \ell_M(Y)) \leq b \quad (615)$$

Remark. These generalize bounds in part I where Y was required to be connected. Bounds of this type were used extensively in the papers of Gawedski and Kupiainen, see for example [21].

Proof.

1. We take $M = 1$ and drop the subscript M . The sum is dominated by

$$\sum_{n=0}^{\infty} \sum_{\{\square_1, \dots, \square_n\}} \exp(-a' \ell'(\square_0 \cup \square_1 \cup \dots \cup \square_n)) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\square_1, \dots, \square_n)} \exp(-a' \ell'(\square_0 \cup \square_1 \cup \dots \cup \square_n)) \quad (616)$$

where the sum is first over unordered collections of distinct blocks and then over ordered collections of distinct blocks.

For every $(\square_1, \dots, \square_n)$ there is at least one tree τ on $(0, 1, 2, \dots, n)$ such that the induced length

$$d_\tau(\square_0, \square_1, \dots, \square_n) \equiv \sum_{\{i, j\} \in \tau} d'(\square_i, \square_j) \quad (617)$$

satisfies $d_\tau(\square_0, \square_1, \dots, \square_n) = \ell'(\square_0, \square_1, \dots, \square_n)$. Here $d'(\square, \square')$ is the distance between centers. Thus our sum is dominated by

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\tau} \sum_{\substack{(\square_1, \dots, \square_n): \\ d_\tau(\square_0, \square_1, \dots, \square_n) = \ell'(\square_0, \square_1, \dots, \square_n)}} \exp\left(-a' d_\tau(\square_0, \square_1, \dots, \square_n)\right) \\ & \leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\tau} \sum_{(\square_1, \dots, \square_n)} \prod_{\{i, j\} \in \tau} \exp\left(-a' d'(\square_i, \square_j)\right) \end{aligned} \quad (618)$$

In the second step we dropped the restriction of the sum over $(\square_1, \dots, \square_n)$. Now we sum over the outer leaves of the tree working our way back to the root at 0, using the bound

$$\sum_{\square': \square' \neq \square} \exp\left(-a' d'(\square, \square')\right) \leq \mathcal{O}(1) e^{-a'} \quad (619)$$

Then the expression is dominated by

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\mathcal{O}(1) e^{-a'}\right)^n \sum_{\tau} 1 \leq \sum_{n=0}^{\infty} (\mathcal{O}(1) e^{-a'})^n \leq \mathcal{O}(1) \quad (620)$$

for a' sufficiently large. Here we use Cayley's formula that there are n^{n-2} tree graphs on n vertices. Here this is $(n+1)^{n-1} \leq \mathcal{O}(1) n!$.

2. Using the second and third bound in lemma E.1 we have for a, b sufficiently large

$$\begin{aligned} \sum_{Y: Y \supset \square_0} \exp(-a \ell(Y)) & \leq \sum_{Y: Y \supset \square_0} \exp\left(-(a - a') \ell(Y) + a' \sqrt{d} |Y| - a' \ell'(Y)\right) \\ & \leq \mathcal{O}(1) \sum_{Y: Y \supset \square_0} \exp\left(-(a - \mathcal{O}(1)) \ell(Y)\right) \exp(-a' \ell'(Y)) \\ & \leq \mathcal{O}(1) \sum_{Y: Y \supset \square_0} \exp(-a' \ell'(Y)) \leq b \end{aligned} \quad (621)$$

Lemma E.3. *Let Ω be a union of M -cubes. For $\square \subset \Omega$ and constants $\kappa_0, K_0 = \mathcal{O}(1)$*

$$\sum_{X \in \mathcal{D}_k(\text{mod } \Omega^c), X \supset \square} \exp\left(-\kappa_0 d_M(X, \text{mod } \Omega^c)\right) \leq K_0 \quad (622)$$

Proof. We have $d_M(X, \text{mod } \Omega^c) \geq \tilde{\ell}_M(X \cap \Omega) \geq \frac{1}{2} \ell_M(X \cap \Omega)$. Thus it suffices to show

$$\sum_{X \in \mathcal{D}_k(\text{mod } \Omega^c), X \supset \square} \exp\left(-\frac{1}{2} \kappa_0 \ell_M(X \cap \Omega)\right) \leq K_0 \quad (623)$$

We classify the terms in the sum by the $Y = X \cap \Omega$ they generate. Although X is connected, Y need not be. The sum can then be written

$$\sum_{Y: Y \supset \square} \exp\left(-\frac{1}{2} \kappa_0 \ell_M(Y)\right) \left| \{X \in \mathcal{D}_k(\text{mod } \Omega) : X \cap \Omega = Y\} \right| \quad (624)$$

Thus we must estimate the number of polymers $X \in \mathcal{D}_k(\text{mod } \Omega)$ such that $X \cap \Omega = Y$. If $\{\Omega_\alpha^c\}$ are the connected components of Ω^c , then any such X can be written

$$X = Y \cup \left(\bigcup_\alpha (X \cap \Omega_\alpha^c) \right) \quad (625)$$

By our assumptions either $X \cap \Omega_\alpha^c = \emptyset$ or $X \cap \Omega_\alpha^c = \Omega_\alpha^c$. Since X is connected each non-empty $X \cap \Omega_\alpha^c$ must have a block sharing a face with a block in Y . Thus the number of non-empty $X \cap \Omega_\alpha^c$ is at most $2^d |Y|_M$. Counting the number of X 's generating a particular Y means choosing a subset of this set. Thus there are less than $2^{2^d |Y|_M}$ such X . Now our sum is bounded by

$$\sum_{Y: Y \supset \square} \exp \left(-\frac{1}{2} \kappa_0 \ell_M(Y) + \mathcal{O}(1) |Y|_M \right) \quad (626)$$

But $|Y|_M \leq \mathcal{O}(1) \ell_M(Y) + \mathcal{O}(1)$ by lemma E.1 and so if κ_0 is large enough the sum is bounded by

$$\mathcal{O}(1) \sum_{Y: Y \supset \square} \exp \left(-\frac{1}{4} \kappa_0 \ell_M(Y) \right) \quad (627)$$

The result now follows by lemma E.2 provided $\kappa_0 > 4a$.

F cluster expansion with holes

We quote a special version of the cluster expansion in which there are holes for the large field region. See also [16], [18].

On a unit lattice consider subsets Ω, Λ which are unions of M -cubes, and satisfy $\Lambda \subset \Omega$. we consider integrals of the form

$$\Xi = \int \exp \left(\sum_{X \in \mathcal{D}_k(\text{mod } \Omega^c), X \cap \Lambda \neq \emptyset} H(X, \Phi', \Phi) \right) d\mu_\Lambda(\Phi) \quad (628)$$

Here μ_Λ is an ultralocal probability measure on the fields $\Phi : \Lambda \rightarrow \mathbb{R}$ rendering them independent random variables. The Φ' are any other fields, and $H(X, \Phi', \Phi)$ depends on Φ', Φ only in X .

Theorem F.1. *Let $c_0 = \mathcal{O}(1)$ be sufficiently small, let $H_0 \leq c_0$, let $\kappa \geq 3\kappa_0 + 3$, and suppose*

$$|H(X, \Phi', \Phi)| \leq H_0 e^{-\kappa d_M(X, \text{mod } \Omega^c)} \quad X \in \mathcal{D}_k(\text{mod } \Omega^c) \quad (629)$$

on the support of μ_Λ . Then

$$\Xi = \exp \left(\sum_{Y \in \mathcal{D}_k(\text{mod } \Omega^c), Y \cap \Lambda \neq \emptyset} H^\#(Y, \Phi') \right) \quad (630)$$

where $H^\#(Y, \Phi')$ depends on Φ' only in Y and satisfies

$$|H^\#(Y, \Phi')| \leq \mathcal{O}(1) H_0 e^{-(\kappa - 3\kappa_0 - 3)d_M(Y, \text{mod } \Omega^c)} \quad (631)$$

Remark. In addition $H^\#(Y)$ only depends on $H(X)$ for $X \subset Y$. We call this the *local influence* property of the cluster expansion.

Proof. This closely follows the proof of the standard cluster expansion. It is exposed in Appendix B in part I, to which we refer for more details. The differences are that instead of general polymers X , we have polymers $X \in \mathcal{D}_k(\text{mod } \Omega^c)$ with holes Ω^c , and instead of decay rates $e^{-\kappa d_M(X)}$ we have decay rates $e^{-\kappa d_k(X, \text{mod } \Omega^c)}$ outside the holes. Key ingredients are the bound

$$\sum_{Y: Y \cap Y' \cap \Omega \neq \emptyset} e^{-\kappa_0 d_M(Y, \text{mod } \Omega^c)} \leq \mathcal{O}(1) |Y' \cap \Omega|_M \quad (632)$$

from (622) and the bound

$$|Y' \cap \Omega|_M \leq \mathcal{O}(1) (d_M(Y', \text{mod } \Omega^c) + 1) \quad (633)$$

We have stated these estimates in a form that takes into account that there is a modified notion of connectedness between polymers. Now we say that $X_1, X_2 \in \mathcal{D}_k(\text{mod } \Omega^c)$ are Ω -connected if $X_1 \cap \Omega$ and $X_2 \cap \Omega$ have non-empty intersection (i.e if $X_1 \cap X_2 \cap \Omega \neq \emptyset$). Otherwise $X_1 \cap \Omega$ and $X_2 \cap \Omega$ have empty intersection and they are called Ω -disjoint. Note that Ω -connected implies connected, but Ω -disjoint does not imply disjoint.

We sketch some details of the proof. First we make a Mayer expansion and write

$$\exp \left(\sum_X H(X) \right) = \sum_{\{Y_j\}} \prod_j K(Y_j) \quad (634)$$

where the Y_j are Ω -disjoint, and where where for $Y \subset \mathcal{D}_k(\text{mod } \Omega^c)$ and $Y \cap \Lambda \neq \emptyset$.

$$K(Y) = \sum_{\{X_i\}: \cup_i X_i = Y} \prod_i (e^{H(X_i)} - 1) \quad (635)$$

The latter sum is restricted by the condition that the X_i are Ω -connected, i.e. cannot be divided into Ω -disjoint sets. $K(Y) = K(Y, \Phi', \Phi)$ only depends on Φ' in Y and Φ in $Y \cap \Lambda$.

To estimate $K(Y)$ we need to show that if $\{X_i\}$ has n elements

$$d_M(Y, \text{mod } \Omega^c) \leq \sum_i d_M(X_i, \text{mod } \Omega^c) + (n - 1) \quad (636)$$

To see this let τ_i be minimal graphs on the cubes in $X_i \cap \Omega$ of length $\ell(\tau_i) = M d_M(X_i, \text{mod } \Omega^c)$. Stitch together the τ_i to get a graph τ on the cubes in $\cup_i (X_i \cap \Omega) = Y \cap \Omega$ with $\ell(\tau) \leq \sum_i \ell(\tau_i) + M(n - 1)$. Since $M d_M(Y, \text{mod } \Omega^c) \leq \ell(\tau)$ this gives the result.

With some further analysis the bounds (632), (633), (636) lead to the bound on the support of μ_Λ

$$|K(Y)| \leq \mathcal{O}(1) H_0 e^{-(\kappa - \kappa_0 - 2) d_M(Y, \text{mod } \Omega^c)} \quad (637)$$

Because the $Y_j \cap \Omega$ are disjoint, the $Y_j \cap \Lambda$ are disjoint. Since $K(Y, \Phi', \Phi)$ depends on Φ only in $Y_i \cap \Lambda$, and because fields at different sites are independent random variables

$$\int \left(\sum_{\{Y_j\}} \prod_j K(Y_j, \Phi', \Phi) \right) d\mu_\Lambda(\Phi) = \sum_{\{Y_j\}} \prod_j K^\#(Y_j, \Phi') \quad (638)$$

where

$$K^\#(Y, \Phi') = \int K(Y, \Phi', \Phi) d\mu_\Lambda(\Phi) \quad (639)$$

again satisfies the bound (637).

Next we exponentiate the sum and get

$$\sum_{\{Y_j\}} \prod_j K^\#(Y_j) = \exp \left(\sum_Y H^\#(Y) \right) \quad (640)$$

where for $Y \subset \mathcal{D}_k(\text{mod } \Omega^c)$ and $Y \cap \Lambda \neq \emptyset$.

$$H^\#(Y) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Y_1, \dots, Y_n): \cup_i Y_i = Y} \rho^T(Y_1, \dots, Y_n) \prod_i K^\#(Y_i) \quad (641)$$

and $\rho^T(Y_1, \dots, Y_n)$ is now defined so it vanishes if the Y_j are not Ω -connected.

Finally, with some further analysis, the bound on $K^\#(Y)$ and the estimates (632), (633), (636), (637) lead to the convergence of the series and the estimate (631).

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